## 4.3 HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH

**4.3** | #s 3, 4, 6, 8, 11, 14, 17, 18, 32, 39, 41, 52, 56

#s 20 – 28 are standard test-type questions that are easy, if you have the concept(s) and impossible if you don't.

**INCREASING/DECREASING TEST** BEWARE THE CONVERSE

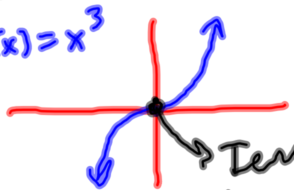
(a) If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

Find the intervals on which $f$ is increasing or decreasing.

$f$ is increasing on $(a,b)$ if, whenever $c < d$ are in $(a,b)$, $f(c) < f(d)$

$f(x) = x^3$



Terrace: $f' = 0$, but $f$ is increasing.

$f'(x) = 3x^2 \overset{SET}{=} 0 \Rightarrow x = 0.$

So $f'(0) = 0$ is not greater than zero, BUT $f$ is increasing on its domain.

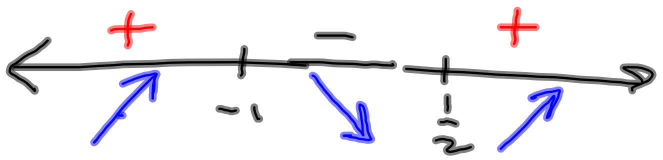
**10.** $f(x) = 4x^3 + 3x^2 - 6x + 1$

$f'(x) = 12x^2 + 6x - 6 \overset{\text{SET}}{=} 0$

$6(2x^2 + x - 1) = 0$

$\Rightarrow (2x - 1)^1 (x + 1)^1 = 0$

$x = \frac{1}{2}, -1$




Increasing : $(-\infty, -1) \cup (\frac{1}{2}, \infty)$

Decreasing : $(-1, \frac{1}{2})$

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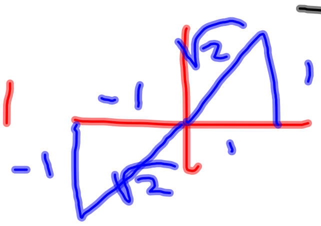
13. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

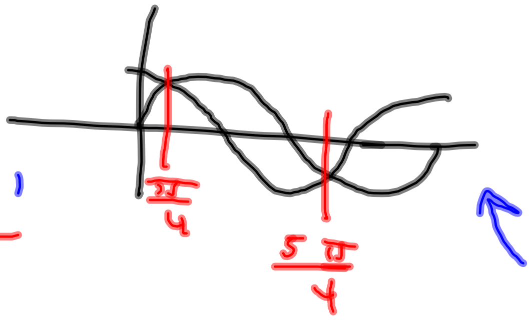
$f'(x) = \cos x - \sin x \overset{\text{SET}}{=} 0$

$\cos x = \sin x$

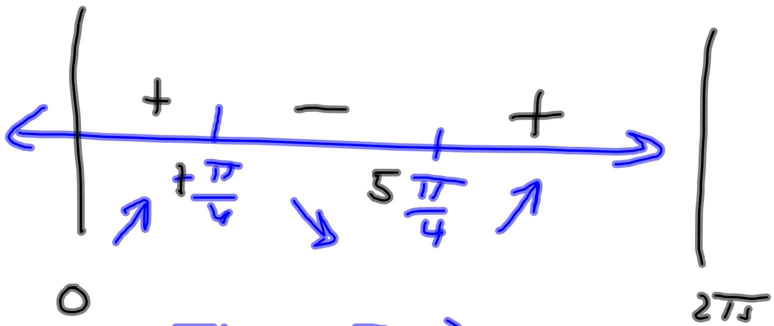
$\frac{\cos x}{\sin x} = 1$

$\cot x = 1$




$x = \frac{\pi}{4}, \frac{5\pi}{4}$




Increasing: $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi)$

Decreasing: $(\frac{\pi}{4}, \frac{5\pi}{4})$

Briefly: What does " $c$ is a critical number of a continuous function $f$ " mean?

THE FIRST DERIVATIVE TEST Suppose that $c$ is a critical number of a continuous function $f$.

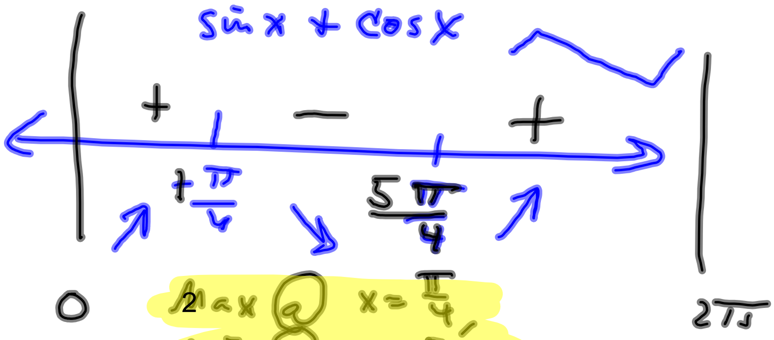
(a) If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.

(b) If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.

(c) If $f'$ does not change sign at $c$ (for example, if $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local maximum or minimum at $c$.

* $x = \pm \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$




Max @ $x = \frac{\pi}{4}$

Min @ $x = \frac{5\pi}{4}$

→ Locals.

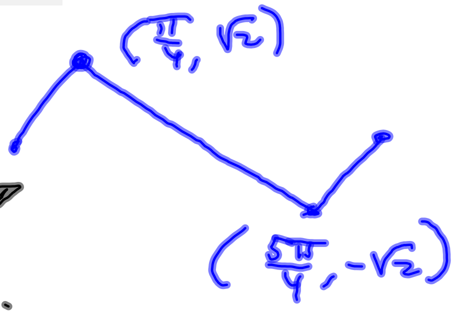
(b) Find the local maximum and minimum values of $f$.

**13.** $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$f(\frac{5\pi}{4}) = -\sqrt{2}$

we have this much from the derivative. →



**DEFINITION** If the graph of $f$ lies above all of its tangents on an interval $I$, then it is called **concave upward** on $I$. If the graph of $f$ lies below all of its tangents on $I$, it is called **concave downward** on $I$.

Positive Attitude

**CONCAVITY TEST**

(a) If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.

(b) If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward on $I$.

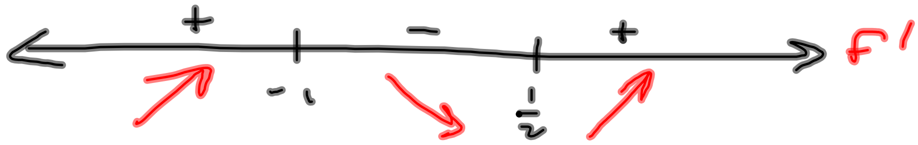
Negative Attitude

Find the intervals of concavity

**10.** $f(x) = 4x^3 + 3x^2 - 6x + 1$

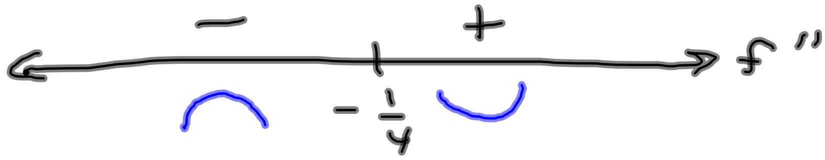
$f'(x) = 12x^2 + 6x - 6 \overset{\text{SET}}{=} 0$

$\Longrightarrow x = \frac{1}{2}, -1$




$f''(x) = 24x + 6 \overset{\text{SET}}{=} 0$

$6(4x+1) = 0 \Longrightarrow x = -\frac{1}{4}$




Sketch of f(x) from this info.

$f(-1) = 6$, $f(-\frac{1}{4}) = 4(-\frac{1}{4})^3 + 3(-\frac{1}{4})^2 - 6(-\frac{1}{4}) + 1$

$= -\frac{1}{16} + \frac{3}{16} + \frac{3}{2} + \frac{16}{16} = \frac{-1+3+24+16}{16}$

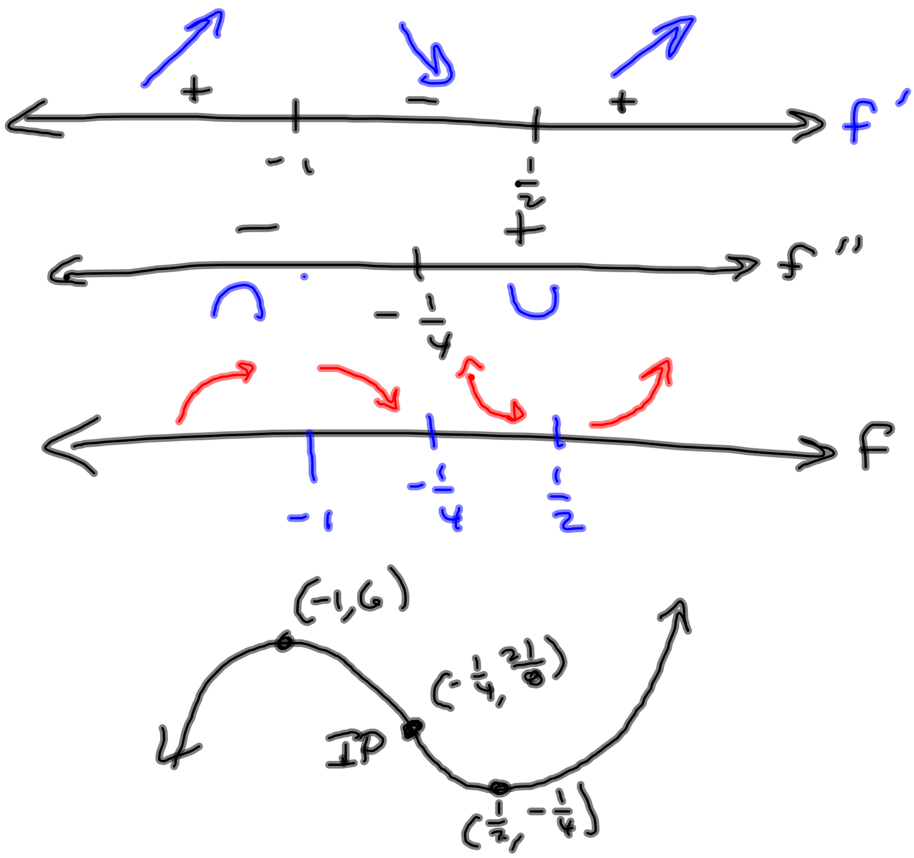
$= \frac{42}{16} = \frac{21}{8} = f(-\frac{1}{4})$

$f(\frac{1}{2}) = 4(\frac{1}{2})^3 + 3(+\frac{1}{2})^2 - 6(+\frac{1}{2}) + 1$

$= 4(\frac{1}{8}) + \frac{3}{4} - 3 + 1$

$= \frac{2}{4} + \frac{3}{4} - \frac{8}{4} = -\frac{3}{4} = f(\frac{1}{2})$

0



**DEFINITION** A point $P$ on a curve $y = f(x)$ is called an **inflection point** if $f$ is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at $P$.

Since the second derivative is to the first derivative just as the first derivative is to the original function, what can you say about the graph of the first derivative at an inflection point?

Find the inflection points.

**10.** $f(x) = 4x^3 + 3x^2 - 6x + 1$

**13.** $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

**THE SECOND DERIVATIVE TEST** Suppose $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

**19.** Suppose $f''$ is continuous on $(-\infty, \infty)$.

(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about $f$?

(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about $f$?

Mar 2-9:54 AM

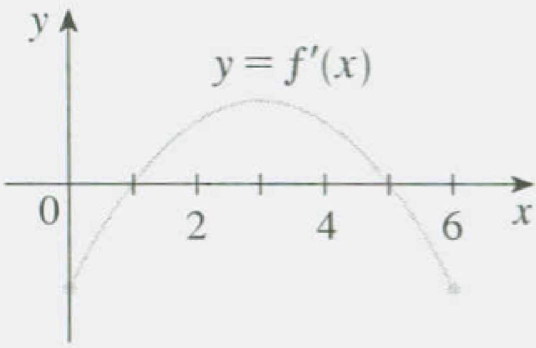
5–6 The graph of the *derivative* $f'$ of a function $f$ is shown.

(a) On what intervals is $f$ increasing or decreasing?

(b) At what values of $x$ does $f$ have a local maximum or minimum?

At what values of $x$ does $f$ have an inflection point?

5.

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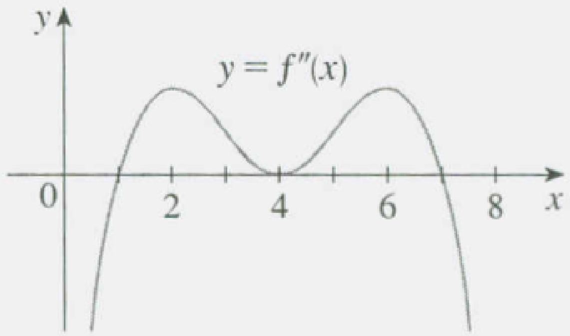
7. The graph of the second derivative $f''$ of a function $f$ is shown. State the $x$-coordinates of the inflection points of $f$. Give reasons for your answers.



9–14

(a) Find the intervals on which $f$ is increasing or decreasing.
(b) Find the local maximum and minimum values of $f$.
(c) Find the intervals of concavity and the inflection points.

We did these 3 things for #s 10 and 13. The way _I_ learned to use derivatives to graph was to analyze the sign patterns of the first and second derivatives on a number line, stack these number lines, and do a 3rd number line that incorporates the information from BOTH into the graph. Just from these analyses, I was able to "see" the graph and finish a clean version of it.

All these pieces we've been putting together are aimed at putting together graphs that show all extrema and inflection points - at really capturing the "shape" of these functions, thru calculus.

**10.** $f(x) = 4x^3 + 3x^2 - 6x + 1$

Mar 2-10:00 AM

20–25 Sketch the graph of a function that satisfies all of the given conditions.

21. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$

27–28 The graph of the derivative $f'$ of a continuous function $f$ is shown.

(a) On what intervals is $f$ increasing or decreasing?

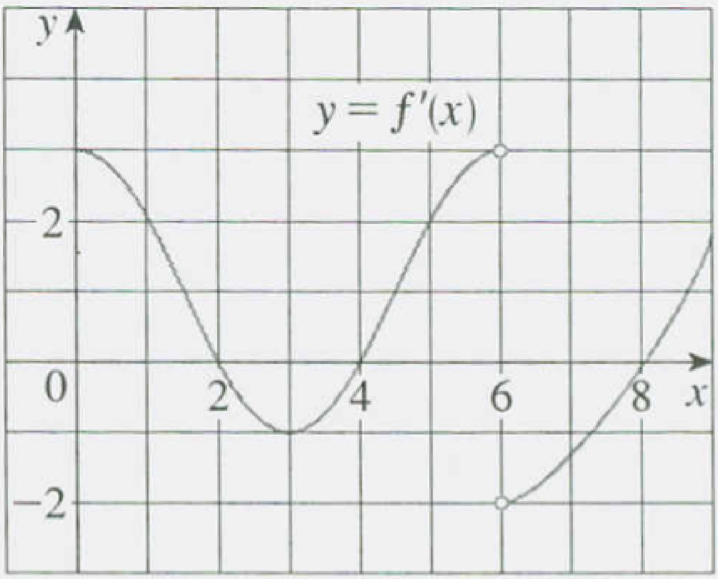
(b) At what values of $x$ does $f$ have a local maximum or minimum?

(c) On what intervals is $f$ concave upward or downward?

(d) State the $x$-coordinate(s) of the point(s) of inflection.

(e) Assuming that $f(0) = 0$, sketch a graph of $f$.

27.



29–40

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work with a graphing device if you have one.

You may want to bookmark the following link in your web browser. It can come in handy.

**http://people.hofstra.edu/stefan_waner/realworld/functions/func.html**

**29.** $f(x) = 2x^3 - 3x^2 - 12x$

43–44

(a) Use a graph of $f$ to estimate the maximum and minimum values. Then find the exact values.

(b) Estimate the value of $x$ at which $f$ increases most rapidly. Then find the exact value.

43. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

Mar 2-10:48 AM

59. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

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