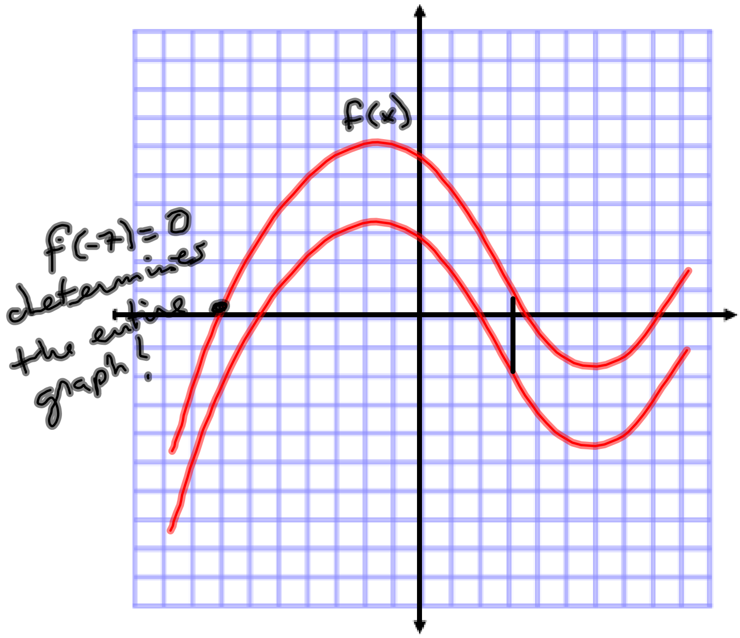
MVT can be used to prove the following, which seems like a pretty obvious result.

$\boxed{5}$ THEOREM If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

And the result above leads very smoothly to the following corollary:

$\boxed{7}$ COROLLARY If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

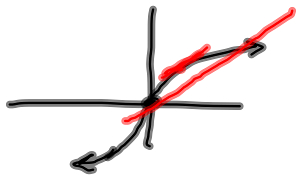


All we need to totally determine the graph of a differentiable function is its derivative and its height at ONE POINT.

Before we start applying Rolle's and MVT to the exercises, note that there is no prescription for *finding* the value c. These theorems simply say that such a c **exists**. This *is* helpful, but only insofar as we know that there *is* a solution, so we're not wasting our time looking for something that isn't there...

(4) $f(x) = \cos(2x)$ on $\left[\frac{\pi}{8}, \frac{7\pi}{8}\right]$

(13) $f(x) = \sqrt[3]{x}$ on $[0,1]$



cnt^{S} $\forall x \in \mathbb{R} \supset [0,1]$ ✓

difb^{l} $\forall x \in \underbrace{\mathbb{R} \setminus \{0\}}_{(-\infty,0) \cup (0,\infty)} \supset (0,1)$

$f(0) = 0$, $f(1) = 1$

$$m_{avg} = \frac{f(b) - f(a)}{b-a} = \frac{f(1) - f(0)}{1-0} = \frac{1-0}{1} = 1$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \overset{\text{SET}}{=} 1$$

$$\frac{1}{3x^{\frac{2}{3}}} = 1$$

$$1 = 3x^{\frac{2}{3}}$$

$$x^{2/3} = \frac{1}{3}$$

$$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left(\frac{1}{3}\right)^{\frac{3}{2}}$$

$$x = \left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{\sqrt{3}}{9}$$

Steve lost a solution with the $\frac{3}{2}$-power thingie.

$$\left(x^{\frac{2}{3}}\right)^3 = \left(\frac{1}{3}\right)^3$$

$$x^2 = \frac{1}{3^3}$$

$$x = \pm\sqrt{\frac{1}{3^3}} \Longrightarrow \boxed{c = \frac{\sqrt{3}}{9}}$$

$$\sqrt[2]{\frac{1}{27}} = \sqrt{\frac{1}{9} \cdot \frac{1}{3}} = \frac{1}{3}\sqrt{\frac{1}{3}} = \frac{1}{3}\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{9}$$

#23 If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ be?

Consider the line thru $(1,10)$ with slope $m = 2$.

$$y = m(x - x_1) + y_1$$
$$= 2(x-1) + 10 \longrightarrow$$
$$y\big|_{x=4} = 2(4-1) + 10 = 16$$

$$10 + 2(4-1) = 16$$

$$\frac{y - y_1}{x - x_1} = m$$

$$\frac{y - 10}{4 - 1} = 2$$

$$\boxed{f(4) \geq 16}$$

#29 Use MVT to prove

$|\sin a - \sin b| \le |a-b|$

Hmmmmmmm

$$\left|\frac{\sin a - \sin b}{a-b}\right| \le 1$$

case $a=b$

Degenerate

$|\sin a - \sin a| = 0 = |a-b|$ ✓

Now assume $a \ne b$. $\boxed{\text{wlog } a<b}$

"without loss of generality"

$\frac{\sin a - \sin b}{a-b}$ looks like m_{avg} for $\sin x$ on $[a,b]$

MVT says ... $\frac{\sin a - \sin b}{a-b} = \cos(c)$ for

some $c \in (a,b)$

So, if $a \ne b$

$$\left|\frac{\sin a - \sin b}{a-b}\right| = |\cos c| \le 1$$

$\Rightarrow |\sin a - \sin b| = |\cos c|\,|a-b| \le |a-b|$

Proof

The degenerate case $a=b$:

Then $|\sin a - \sin b| = 0 = |a-b|$, so the conclusion holds.

Now assume $a \neq b$ (and wlog $a < b$)

Then $\left|\frac{\sin a - \sin b}{a-b}\right| = |\cos c| \leq 1$, for some $c \in (a,b)$, by MVT. This implies

$|\sin a - \sin b| \leq |a-b|$ ∎

(24) S $3 \le \boxed{f'(x) \le 5} \quad \forall x.$

Show that $\underline{18 \le f(8) - f(2) \le 30}$

$f(2) + 5(8-2)$

$f(2) + 3(8-2)$
$= f(2) + 18$

$$\frac{f(8)-f(2)}{8-2} = f'(c)$$

for some $c \in (2, 8)$

$\therefore$

$$3 \le \frac{f(8)-f(2)}{6} = f'(c) \le 5$$

$\Rightarrow \quad 18 \le f(8) - f(2) \le 30$

Given:

$$3 \le f'(x) \le 5$$

__Claim:__ $18 \le f(8) - f(2) \le 30$

__Proof__

$$\frac{f(8)-f(2)}{8-2} = \underbrace{f'(c) \text{ for some } c \in (2,8)}_{\text{MVT.}}$$

This implies

$$3 \le \frac{f(8)-f(2)}{6} \le 5 \implies$$

$$18 \le f(8) - f(2) \le 30 \quad \blacksquare$$

1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

3. $f(x) = \sqrt{x} - \frac{1}{3}x, \quad [0, 9]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

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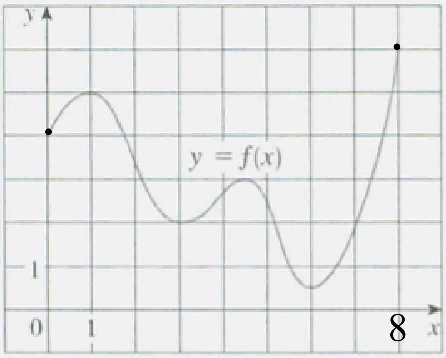
7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

14. $f(x) = \dfrac{x}{x+2}$, $[1, 4]$

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15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

Suppose $f(3) = 2$ and $f'(x) \leq 4$ for all $x \in [-1, 11]$. Give an upper bound for $f(6)$. (Cf #23)

Feb 27-9:40 AM

Suppose $2 \le f'(x) \le 7$ for all x. Prove that $8 \le f(5) - f(1) \le 28$.
(Cf #24).

Intuition Guide:
I always visualize MVT in terms of slope. So, assuming the hypotheses of MVT are satisfied, I think of the conclusion this way:

$$\exists c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$$

But for practical purposes, and in the exercises, we often see the conclusion written this way:

$$\exists c \in (a,b) \ni f(b) - f(a) = f'(c)(b - a)$$

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint*: Apply the Mean Value Theorem to the function $h = f - g$.]

Feb 27-10:38 AM