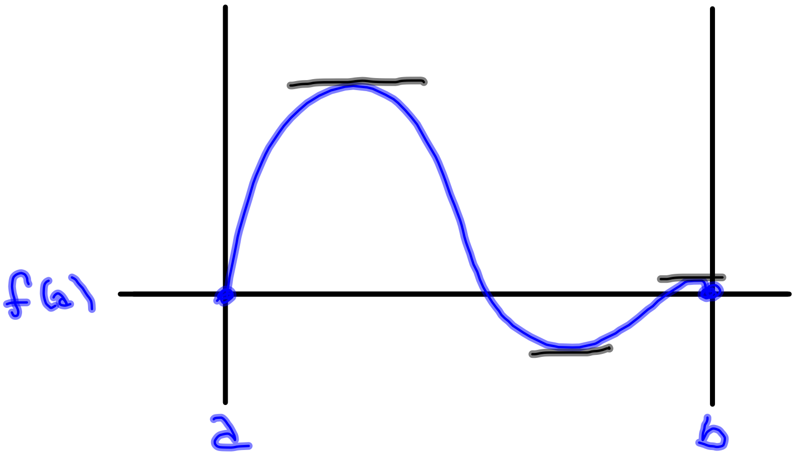
4.2 THE MEAN VALUE THEOREM

ROLLE'S THEOREM Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



THE MEAN VALUE THEOREM Let f be a function that satisfies the following hypotheses:

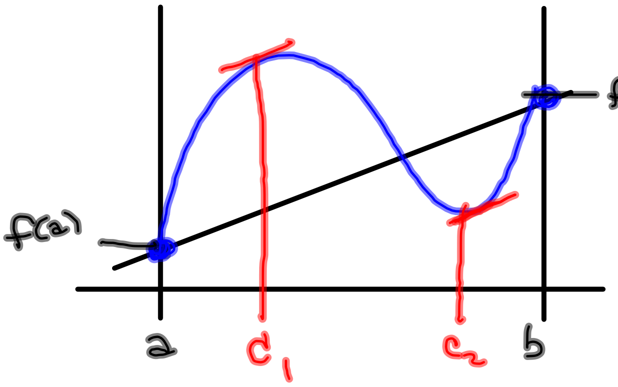
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

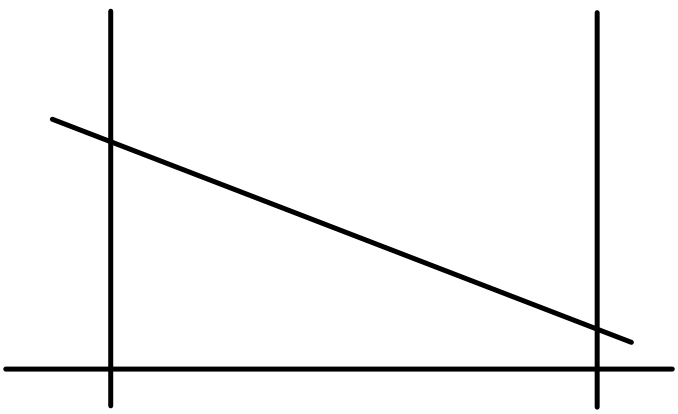
$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$



MVT <u>is</u> Rolle's Theorem with different heights (possibly) at the endpoints.



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MVT can be used to prove the following, which seems like a pretty obvious result.

5 **THEOREM** If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

And the result above leads very smoothly to the following corollary:

7 **COROLLARY** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Before we start applying Rolle's and MVT to the exercises, note that there is no prescription for *finding* the value c. These theorems simply say that such a c **exists**. This *is* helpful, but only insofar as we know that there *is* a solution, so we're not wasting our time looking for something that isn't there...

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1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

3. $f(x) = \sqrt{x} - \frac{1}{3}x, \quad [0, 9]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

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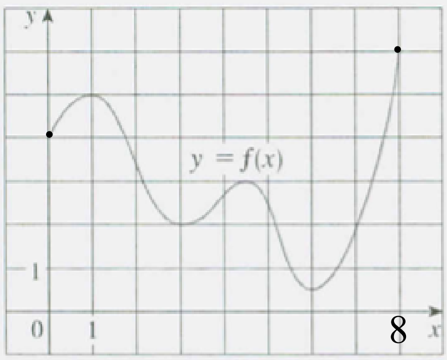
7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

14. $f(x) = \dfrac{x}{x+2}, \quad [1, 4]$

15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

Suppose $f(3) = 2$ and $f'(x) \leq 4$ for all $x \in [-1, 11]$. Give an upper bound for $f(6)$. (Cf #23)

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Suppose $2 \le f'(x) \le 7$ for all x. Prove that $8 \le f(5) - f(1) \le 28$.
(Cf #24).

Intuition Guide:
I always visualize MVT in terms of slope. So, assuming the hypotheses of MVT are satisfied, I think of the conclusion this way:

$$\exists c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$$

But for practical purposes, and in the exercises, we often see the conclusion written this way:

$$\exists c \in (a,b) \ni f(b) - f(a) = f'(c)(b - a)$$

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint*: Apply the Mean Value Theorem to the function $h = f - g$.]

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