**16.** (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius $r$ is measured in micrometers ($1\ \mu m = 10^{-6}$ m). Find the average rate of change of $V$ with respect to $r$ when $r$ changes from

(i) 5 to 8 μm  (ii) 5 to 6 μm  (iii) 5 to 5.1 μm

micron

$$\frac{V(r)-V(5)}{r-5} = \frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi (5)^3}{r-5} = \frac{\frac{4}{3}\pi (r^3 - 5^3)}{r-5} = Y_1$$

$$= \frac{\frac{4}{3}\pi (r-5)(r^2+5r+25)}{r-5} = \frac{4}{3}\pi(r^2+5r+25) = Y_2$$

```
 Plot1 Plot2 Plot3
\Y1=4/3*π*(X^3-5
^3)/(X-5)
\Y2=4/3*π*(X²+5X
+25)
\Y3=
\Y4=
\Y5=
```

```
Y2(8)
        540.3539364
Y2(6)
        381.1799086
Y2(5.1)
        320.4843386
```

```
Y1(8)
        540.3539364
Y1(6)
        381.1799086
Y1(5.1)
        320.4843386
```

Units are

Volume is in $\mu m^3$

radius " " $\mu m$

$\frac{\Delta V}{\Delta r}$ is $\frac{\mu m^3}{\mu m}$

which is $\mu m^2$, but 'ware!

Rate of change in volume w.r.t. change in radius.

(b) Find the instantaneous rate of change of $V$ with respect to $r$ when $r = 5\ \mu m$.

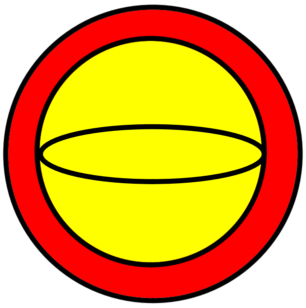
$$V(r) = \frac{4}{3}\pi r^3$$

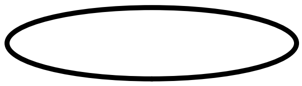
$$V'(r) = \frac{dV}{dr} = 4\pi r^2 = \text{surface area of sphere.}$$

$$V'(5) = 4\pi(5)^2 = 100\pi \approx 314.1592654\ldots \ \mu m^3/\mu m$$

(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

Think of $\Delta V$ as the change in volume





volume of the red part is $\Delta V$
between inner sphere & outer sphere.

Surface area of the outer sphere is
$4\pi r^2$

If we could lay it flat, it'd have
Volume $\underbrace{4\pi r^2}_{\text{Area}} \underbrace{\Delta r}_{\text{Thickness}} \approx \Delta V$


volume = area of top times height.

So $\frac{\Delta V}{\Delta r} \approx 4\pi r^2$

Insight for later on: Integrals, Areas, Volumes.

Instantaneous Rate of change is $\frac{dV}{dr} = 4\pi r^2$

Examples from text

**EXAMPLE 1** The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where $t$ is measured in seconds and $s$ in meters.

(a) Find the velocity at time $t$. $v(t) = s'(t) = f'(t) = 3t^2 - 12t + 9$

(b) What is the velocity after 2 s? After 4 s? $s'(2) = v(t)$

(c) When is the particle at rest? $v(t) = 0$ Solve.

(d) When is the particle moving forward (that is, in the positive direction)? $(0,1) \cup (3,\infty)$

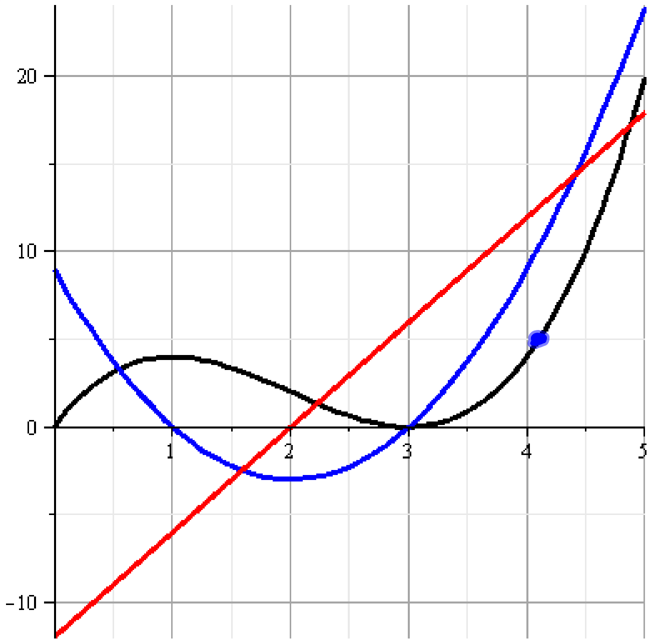
(e) Draw a diagram to represent the motion of the particle.

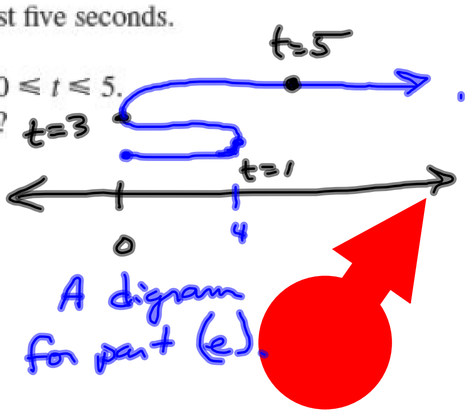
(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time $t$ and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 5$.

(i) When is the particle speeding up? When is it slowing down?




A digram for part (e).

§3.6 #25

$x^2 + xy + y^2 = 3$

Find tangent line @ (1,1)

Diff:

$2x + y + xy' + 2yy' = 0$

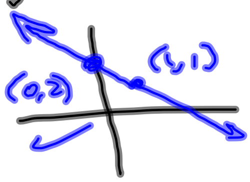
$\vdots$

$y' = \dfrac{-2x-y}{x+2y}$ TABIYAH

$\dfrac{d}{dx}\left[xy\right] = 1 \cdot y + xy'$

product rule; assume y is implicitly a function of x.

$y'\Big|_{\substack{x=1\\ y=1}} = \dfrac{-2-1}{1+2} = -1$




$y = m(x-x_1) + y_1$

$y = -(x-1) + 1$

$\boxed{y = -x+2}$

#12 $\frac{d}{dx}\left[\sin(xy^2)\right] = \cos(xy^2)\left[1\cdot y^2 + x\cdot 2yy'\right]$

$\frac{dy}{dx} = y'$

$\frac{d}{dx}[xy] = y + xy'$

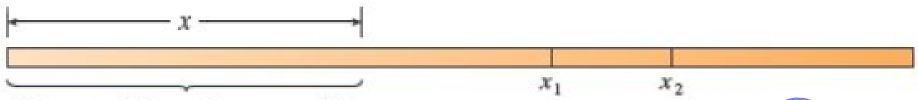
10 $y^5 + x^2y^3 = 1 + x^4y$

$5y^4y' + 2xy^3 + 3x^2y^2y' = 4x^3y + x^4y'$

$5y^4y' + 3x^2y^2y' - x^4y' = 4x^3y - 2xy^3$

$y' = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$

**EXAMPLE 2** If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point $x$ is $m = f(x)$, as shown in Figure 5.




This part of the rod has mass $f(x)$.

$f'(x) = \text{density @ } x$

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\frac{\text{mass}}{\text{unit length}} = \frac{\text{kg}}{\text{m}}$

Density Function @ $x$ is $\frac{dm}{dx} = f'(x)$

**EXAMPLE 3** A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a shaded plane surface. If $\Delta Q$ is the net charge that passes through this surface during a time period $\Delta t$, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** $I$ at a given time $t_1$:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes). □

## BIOLOGY

**EXAMPLE 6** Let $n = f(t)$ be the number of individuals in an animal or plant population at time $t$. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leqslant t \leqslant t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period $\Delta t$ approach 0:

See Examples

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Read the Durn book.

## ECONOMICS

**EXAMPLE 8** Suppose $C(x)$ is the total cost that a company incurs in producing $x$ units of a certain commodity. The function $C$ is called a **cost function**. If the number of items produced is increased from $x_1$ to $x_2$, then the additional cost is $\Delta C = C(x_2) - C(x_1)$,

and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since $x$ often takes on only integer values, it may not make literal sense to let $\Delta x$ approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and $n$ large (so that $\Delta x$ is small compared to $n$), we have

$$C'(n) \approx \boxed{C(n + 1) - C(n)}$$