$$\lim_{x \to 2^-} f(x) = f_-(2)$$

§3.5 stuff

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{d}{dg}\left[f(g)\right] \cdot \frac{dg}{dx}$$

$$\frac{d}{dx}\left[\sqrt{5-x}\right] = \underbrace{\frac{1}{2}(5-x)^{-\frac{1}{2}}}_{\frac{df}{dg}} \cdot \underbrace{(-1)}_{\frac{dg}{dx}}$$

$f(x) = \sqrt{x}$

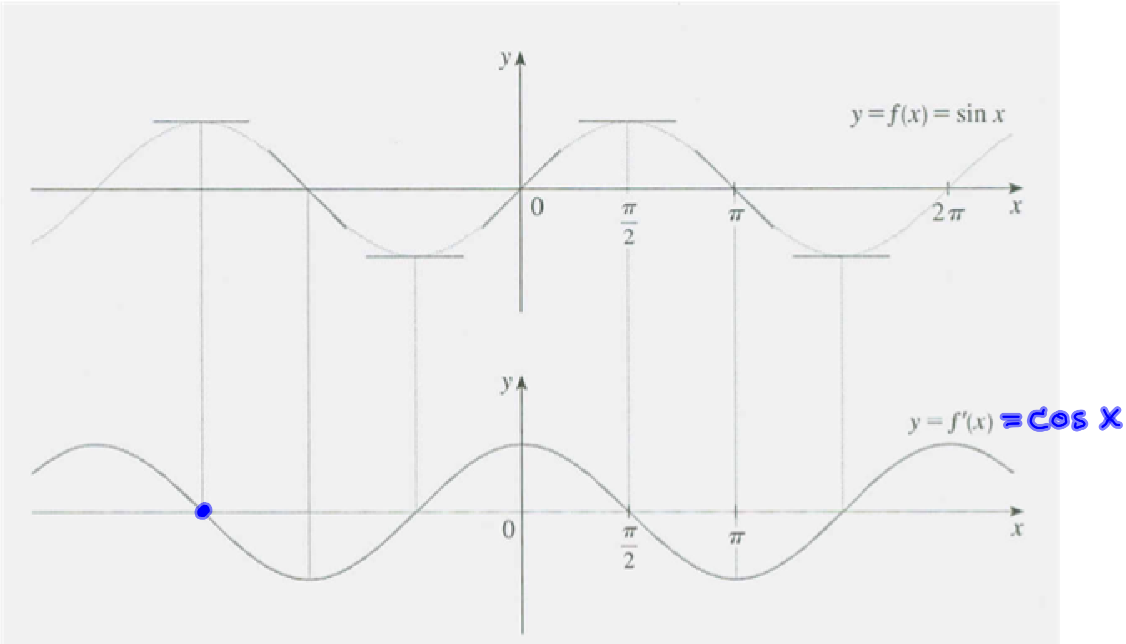
$g(x) = 5-x$

§3.3 #66

$$\frac{dy}{dx} = \frac{d}{dx}\left[y\right] = \frac{d}{dx}\left[\frac{h(x)}{x}\right]$$

3.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Quick Preview of the Derivative of $\sin x$ is given in TEC for 3.4.
http://www.stewartcalculus.com/tec/



I would like to take a little time to *prove* that the derivative of $\sin x$ is actually $\cos x$. To accomplish this, we will need a couple of identities from trigonometry, *and* we will need to get a handle on a couple of limits that will crop up.

Recall:

Angle sum identities (We'll only need the first one for the sine proof. If we were going to do the cosine proof, we'd need the second one, also):

$$\sin(x+h) = \sin x \cos h + \sin h \cos x$$

$$\cos(x+h) = \cos x \cos h - \sin x \sin h$$

The difference quotient for $\sin x$:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}$$
$$= \frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$
$$= \frac{\sin x\cos h - \sin x + \sin h\cos x}{h}$$
$$= \frac{\sin x(\cos h - 1) + \sin h\cos x}{h}$$
$$= \sin x \cdot \frac{\cos h - 1}{h} + \frac{\sin h}{h}\cdot \cos x$$

We wish to pass to the limit as h -> 0. To do this, we will show that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad\qquad \lim_{h\to o}\frac{\cos h - 1}{h} = 0$$

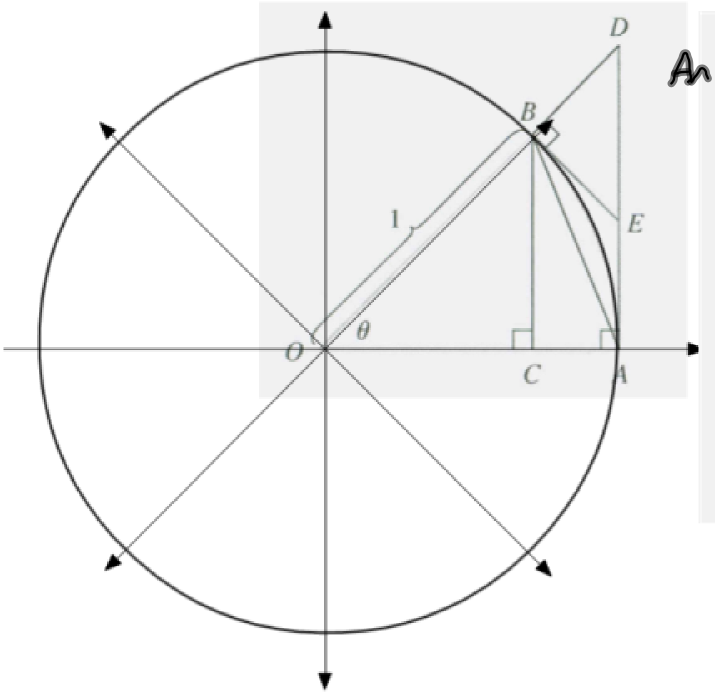
If we can prove that $\lim_{h\to 0}\frac{\sin h}{h} = 1$, we can use it, and some "algebraic trickery" to prove that $\lim_{h\to o}\frac{\cos h - 1}{h} = 0$.

Feb 10-9:12 AM

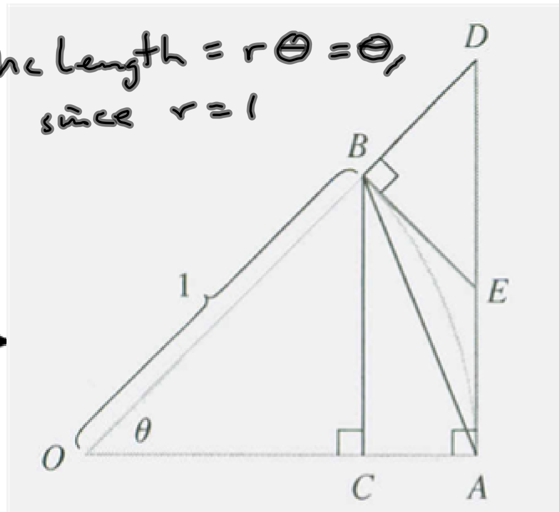
Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Proof: Assume first that $\boxed{\theta \text{ lies between } 0 \text{ and } \pi/2}$.
(Similar arguments hold for negative angles.)

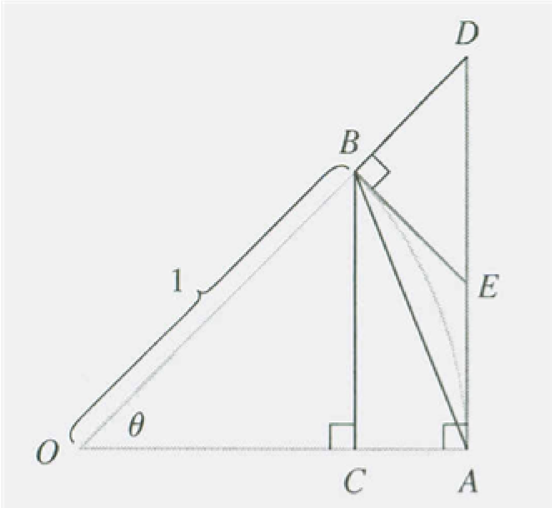
Figure 2(a) shows a sector of a circle with center O, central angle θ, and radius 1



Arc length $= r\theta = \theta$, since $r = 1$



arc $AB = \theta$.



$$\sin\theta = \frac{|BC|}{|OB|} = \frac{|BC|}{1} = |BC| \Rightarrow$$

$$\sin\theta = |BC| < |AB| < \text{arc } AB = \theta$$

in other words,

$$\sin\theta < \theta$$

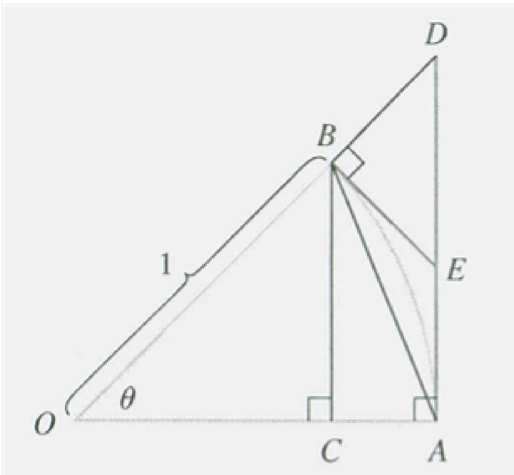
θ lies between 0 and $\pi/2$.

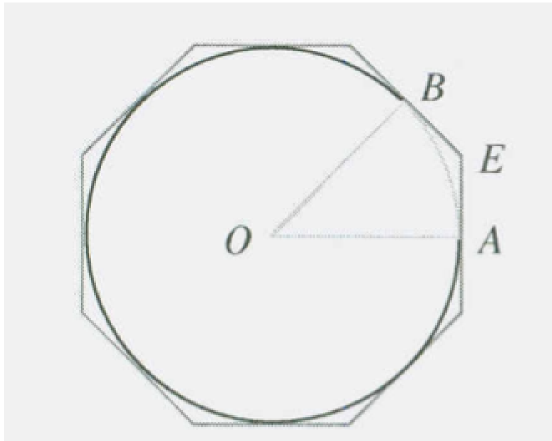
$$\frac{\sin\theta}{\theta} < 1$$

$\frac{\sin\theta}{\theta} < 1$

$? < \frac{\sin\theta}{\theta} < 1$

Want to squeeze $\frac{\sin\theta}{\theta}$ between 1 & 1.





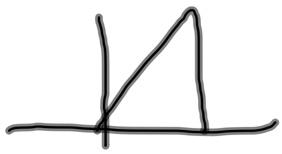
$$\theta = \text{arc } AB < |AE| + |EB|$$

$$< |AE| + |ED|$$

$$= |AD| = |OA| \tan \theta$$

$$= \tan \theta$$

$$\frac{|AD|}{|OA|} = \tan\theta, \text{ so } |AD| = |OA|\tan\theta$$



In other words: $\theta < \tan\theta$

In other words: $\theta < \dfrac{\sin\theta}{\cos\theta}$

So a little algebra gives us

$$\cos\theta < \frac{\sin\theta}{\theta}$$

Combining this with our previous result of $\frac{\sin\theta}{\theta} < 1$

(Cosine and theta are both positive)

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

This compound inequality holds for all θ such that $0 < \theta < \frac{\pi}{2}$

Now we can SQUEEZE it between cosine and 1!

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$2x+3 < 5$ on $(0,1)$ and $\lim_{x\to 1}(2x+3)=5$

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$1 \le \boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1} \le 1$$

By squeeze Thm.

Feb 10-1:09 PM

Claim: $\boxed{\lim_{h \to 0} \frac{\cos h - 1}{h} = 0}$

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}$$

$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

$$= \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$\lim (fg) = \lim f \lim g$

$$= \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$= -\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} \xrightarrow{h \to 0} -1 \cdot \frac{0}{1+1} = 0$$

$\sin^2 h + \cos^2 h = 1$

$1 - \cos^2 h = \sin^2 h$

$\cos^2 h - 1 = -\sin^2 h$

Now we have all the pieces we need to finish our proof:

$$\frac{f(x+h)-f(x)}{h} = \ldots = \sin x \cdot \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cdot \cos x$$

$$\xrightarrow{h \to 0} = (\sin x)(0) + (1)(\cos x) = \cos x$$

$$\boxed{\frac{d}{dx}\left[\sin x\right] = \cos x}$$

#20 prove

$$\frac{d}{dx}\left[\cos x\right] = \sin x$$

But you don't have to prove

The proof for the derivative of the cosine function is very similar, using the angle addition formula for cosine:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\cos(x+h) =$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

Proving that the derivative of cosine is sine is #20 in the exercises. It uses the same sort of argument, only you don't have to do that big, huge geometric argument that I used for the two limits (which crop up again).

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{f'g - fg'}{g^2} = \left(\frac{f}{g}\right)'$$

#19 $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$

#18 $\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right]$

#17 $\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = \boxed{-\frac{\cos x}{\sin^2 x}}$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = \boxed{-\csc x \cot x = \frac{d}{dx}[\csc x]}$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Test Question Differentiate.

$$\frac{d}{dx}\left[\frac{5x^2 \sin x}{\sec x}\right] =$$

$$= \frac{(10x \sin x + 5x^2 \cos x)(\sec x) - (5x^2 \sin x)(\sec x \tan x)}{\sec^2 x} =$$

Feb 9-10:10 AM

$$\frac{d}{dx}\left[\overbrace{5x^2}^{f}\,\overbrace{\sin x\cos x}^{g}\right]$$

Product Rule twice

$$= \underbrace{10x}_{f'}\underbrace{(\sin x\cos x)}_{g} \underbrace{+5x^2}_{+\ f}\underbrace{\left(\cos x\cos x + (\sin x)(-\sin x)\right)}_{g'}$$

g' requires product rule, itself.

$$= \frac{(10x\sin x + 5x^2\cos x)(\sec x) - (5x^2\sin x)(\sec x\tan x)}{\sec^2 x}$$

$$= \frac{10x\sin x\sec x}{\sec^2 x} + \frac{5x^2\cos x\sec x}{\sec^2 x} - \frac{5x^2\sin x\sec x\sin x}{\cos x\cdot\sec^2 x}$$

$$= 10x\sin x\cos x + 5x^2\cos^2 x - 5x^2\sin^2 x$$

$$=$$