37. Prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

$\left[\textit{Hint: } \text{Use } \left| \sqrt{x} - \sqrt{a} \right| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}}. \right]$

This one's more challenging than I intended. Wonder how I missed it.

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

This exercise tries to get at how you would prove a limit does NOT exist.

The POINT, here, is that, no matter how close to 0 you get, there are always points closer to 0 such that $f(x) = 1$ and $f(x) = 0$. This function never settles down to any one value in any neighborhood of 0.

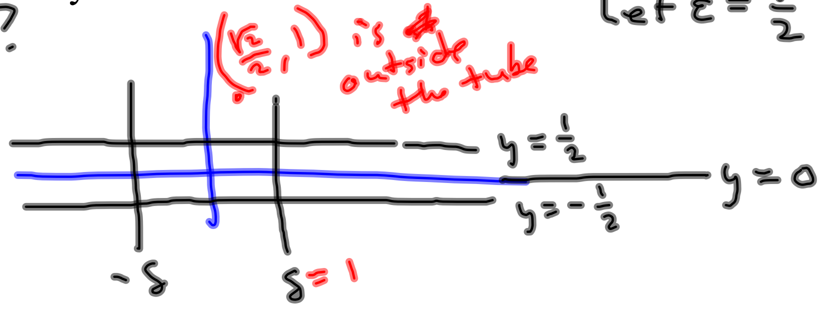
Think about it this way: Since f oscillates between 0 and 1 (with nothing in between), the only choices for $\lim_{x\to 0} f(x)$ are $L = 0$ or $L = 1$. If you try $L = 0$, no matter how close you come to $x = 0$, there's always a point closer to zero where $f(x) = 1$. $\varepsilon = .5$ shoots that one down.

If you try $L = 1$, the same thing happens, with $f(x) = 0$ infinitely many times between any nonzero x-value and 0.

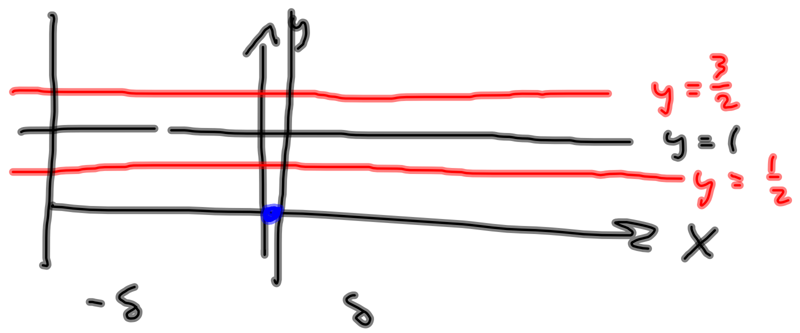
$\lim_{x\to 0} f(x) = 0$?

NO

Let $\varepsilon = \frac{1}{2}$



$\lim_{x\to 0} f(x) = 1$



$\lim_{x \to c} f(x) = L$ means the following:

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

To say that the limit does NOT exist, means that you can find me at least ONE $\varepsilon > 0$ such that *no* $\delta > 0$ can be found that will guarantee that $|f(x) - L| < \varepsilon$ for *all* x-values satisfying $0 < |x - c| < \delta$.

So to do this formally (abstractly), you need to find an actual value of epsilon and show that ANY value of delta that you try won't get the job done.

This is pretty advanced reasoning, that's good for you to try to wrap your head around, but let's remind ourselves of the kinds of test questions I want to write regarding this material.

6. (15 points) Use the precise definition of a limit to prove that $\lim_{x \to 4} (3x - 2) = 10$

Bonus.

I will probably include a $\boxed{\text{quadratic proof}}$ on the test, also, for example, prove that $\lim_{x\to 2}\left(x^2-3x+3\right)=1$. Such a question requires some high-level reasoning.

These quadratics aren't as hard as they seem, assuming you can factor a trinomial!

And these are easy to factor, because when the limit's approaching 2, then the factor $(x - 2)$ will pop out, every time, and *that* factor will correspond to delta. The *hard part* will be to get a bound on the *other* factor. But even that's not as hard as you might think.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x+3)^4} > 10{,}000$$

This is about proving an infinite limit.

To prove that $\frac{1}{(x+3)^4} \xrightarrow{x \to -3} \infty$, then you can make it bigger than ANY fixed value, just by taking x sufficient close to -3.

WANT

$$\frac{1}{(x+3)^4} > 10000$$

$$1 > 10000(x+3)^4$$

$$\frac{1}{10000} > (x+3)^4$$

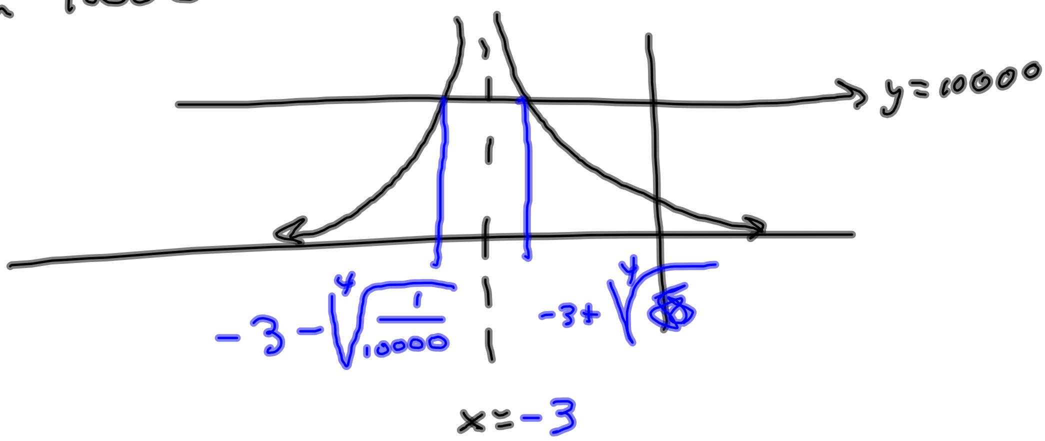
$$\sqrt[4]{\frac{1}{10000}} > \sqrt[4]{(x+3)^4} = |x+3|$$

$$|x+3| < \sqrt[4]{\frac{1}{10000}}$$

Jan 27-9:58 AM

$$|x-(-3)| < \sqrt[4]{\frac{1}{10000}}$$ A bound on the distance from x to -3.

This says, if x is less than $\sqrt[4]{\frac{1}{10000}}$ units from -3, then $\frac{1}{(x+3)^4}$ will be greater than 10000



6 DEFINITION Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

42. Prove, using Definition 6, that $\lim_{x \to -3} \frac{1}{(x+3)^4} = \infty$.

Let $M > 0$ be given. We find $\delta > 0$ so that if $0 < |x - (-3)| < \delta$, we'll have

$\frac{1}{(x+3)^4} > M$. is what we want.

$1 > M(x+3)^4$

$\frac{1}{M} > (x+3)^4$

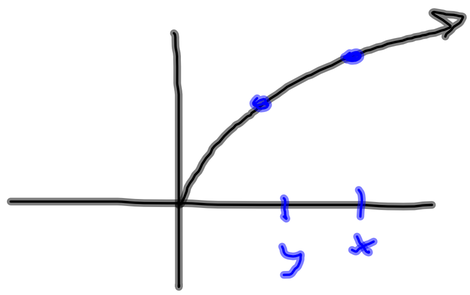
$(x+3)^4 < \frac{1}{M}$

$|x+3| < \sqrt[4]{\frac{1}{M}}$

$x < y \implies \sqrt[4]{x} < \sqrt[4]{y}$

$\sqrt[4]{x}$ is an increasing function

Let $M > 0$ be given, then let $\delta = \sqrt[4]{\frac{1}{M}}$, so, if $0 < |x - (-3)| < \delta$, we have



$$\frac{1}{(x+3)^4} > \frac{1}{\delta^4} = \frac{1}{\left(\sqrt[4]{\frac{1}{M}}\right)^4}$$

$$\frac{1}{3} > \frac{1}{4}$$

$$= \frac{1}{\frac{1}{M}} = M$$

Jan 28-1:12 PM

$$(x-3)(x+2) = x^2 - x - 6 \overset{\text{SET}}{=} 0$$

$$\Longrightarrow x^2 - x - 1 = 5$$

Brain Fart

$$\lim_{x \to 3} (x^2 - x - 1) = 5$$

Want

$$|x^2 - x - 1 - 5| < \varepsilon$$

$$|x^2 - x - 6| < \varepsilon$$

$$|(x-3)(x+2)| < \varepsilon$$

$$\underset{<6}{|x+2|}\,\underbrace{|x-3|}_{<\delta} < \varepsilon$$

If we assume $\delta \leq 1$.

So, $|x+2||x-3| < 6\delta \leq \varepsilon$

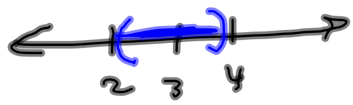
The Key to these quadratics if $x \to 3$, assume it's within 1 unit of $x = 3$, i.e.

$$|x-3| < 1$$

$$\Longrightarrow -1 < x - 3 < 1$$

$$+3 = +3 = +3$$

$$2 < x < 4$$



So how big is $|x+2|$, in this case?

$$2 < x < 4$$

$$2+2 < x+2 < 4+2$$

$$4 < x+2 < 6$$

$$\Longrightarrow |x+2| < 6$$

Claim: $\lim_{x \to 3} (x^2 - x - 1) = 5$

Scratch

want $|x^2 - x - 1 - 5| < \varepsilon$

$|x^2 - x - 6| < \varepsilon$

$\underbrace{|x+2|}\underbrace{|x-3|}_{<\delta} < \varepsilon$

Need a bound on $|x+2|$

$\boxed{\text{Assume } \delta \le 1}$

$-1 \le x - 3 \le 1$

$2 \le x \le 4$

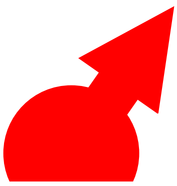
$2+2 \le x+2 \le 4+2$

$4 \le x+2 \le 6$, so

$|x+2| \le 6$

Impatient

Empassioned



$\min\{3, 7\} = 3$

Proof

Let $\varepsilon > 0$ be given.

Define $\delta = \min\{1, \frac{\varepsilon}{6}\}$.

If $0 < |x-3| < \delta$, then

$|x^2 - x - 1 - 5| = |x^2 - x - 6|$

$= |x+2||x-3|$

$\le 6|x-3|$

$< 6\delta$

$\le 6 \cdot \frac{\varepsilon}{6} = \varepsilon$ ■

Test 1 Wednesday,

2.5 Due Wed.

2.5

$\lim_{x \to c} f(x) = f(c)$

is CONTINUITY