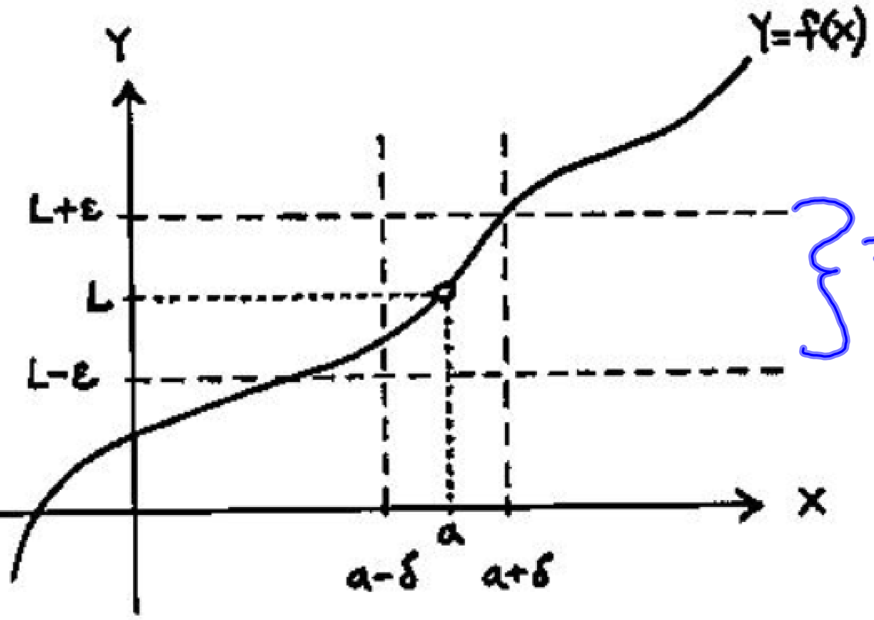


The ε-tube.

2.4 THE PRECISE DEFINITION OF A LIMIT

[2] **DEFINITION** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

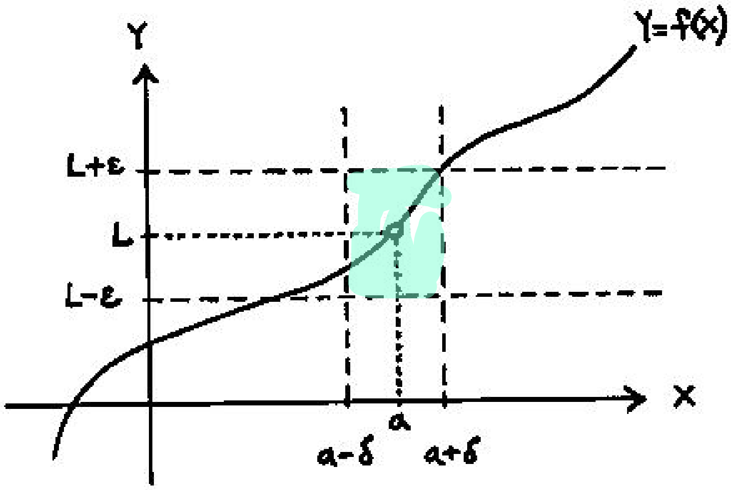
$$\lim_{x \to a} f(x) = L$$

if $\overbrace{\text{for every}}^{\forall}$ number $\varepsilon > 0$ $\overbrace{\text{there is}}^{\exists}$ a number $\delta > 0$ $\overbrace{\text{such that}}^{\ni}$

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

You tell me how close $f(x)$ needs to be to L by giving me the tolerance ε, and I'll tell you how close x needs to be to a by giving you the δ.

$\lim_{x \to a} f(x) = L$ means $\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni$ $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

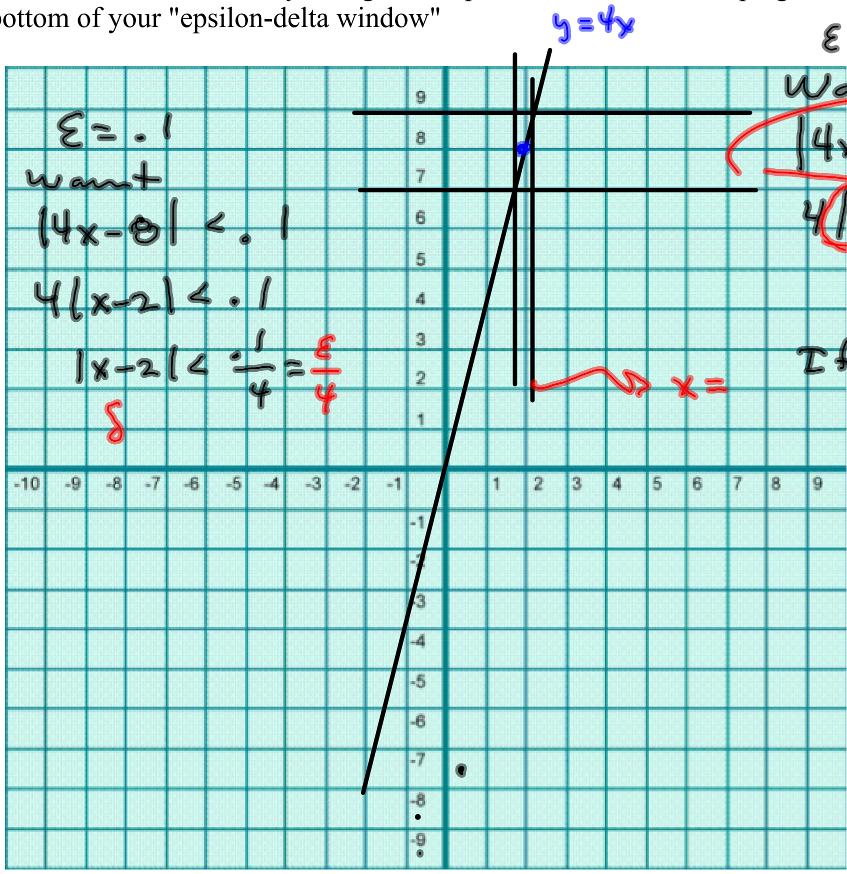


13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.

Oops! This one's assigned!

(c) Repeat part (a) with epsilon just a letter. In other words, prove that the limit as x approaches 2 of $4x$ is 8.

For a linear function, the delta is pretty easy to find, because the SLOPE of the linear function determines how fast the function grows, and if you know how fast it grows, it's easy to make the window skinny enough to keep the function from escaping from the top and bottom of your "epsilon-delta window"



$\varepsilon = 1$ picture

Want

$|4x-8| < 1$

$4|x-2| < 1$

$|x-2| < \frac{1}{4} = \frac{\varepsilon}{4}$

If I choose $\varepsilon = 1$, $\delta = \frac{1}{4}$ will get 'er done.

In general, the δ we associate with ε is $\delta = \frac{\varepsilon}{4}$, for $f(x) = 4x$

what about $f(x) = 13x - 121{,}255$?

"Let $\varepsilon > 0$ be given.

Define $\delta = \frac{\varepsilon}{13}$"

Now that we have δ for any given ε, we write the proof:

<u>Claim</u> $\lim_{x \to 2} (4x) = 8$

<u>Proof</u> Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{4}$.

Then, if $0 < |x-2| < \delta$, we have

$$|f(x) - 8| = |4x - 8| = 4|x-2| < 4\delta$$

$$= \cancel{4} \cdot \frac{\varepsilon}{\cancel{4}} = \varepsilon$$

Prove that $\lim_{x \to 3} (3x-7) = 2$

$|x-3| < \delta$

want: Scratch

$|(3x-7) - 2| < \varepsilon$

$|3x-9| < \varepsilon$

$3\underbrace{|x-3|}_{\delta} < \varepsilon$

Let $3\delta = \varepsilon$

$\delta = \frac{\varepsilon}{3}$

Proof

Let $\varepsilon > 0$ be given.

Define $\delta = \frac{\varepsilon}{3}$. Then,

if $0 < |x-3| < \delta$, we have

$|f(x) - 2| = |(3x-7) - 2|$

$= |3x-9| = 3|x-3| < 3\delta$

$= 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ ∎

This one's easier than it looks...

21. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2} = 5$

22. $\lim_{x \to -1.5} \dfrac{9 - 4x^2}{3 + 2x} = 6$

$\dfrac{x^2+x-6}{x-2} = \dfrac{(x+3)(x-2)}{x-2} = x+3 \quad (x \neq 2)$, so

by the "unnamed theorem", proving #21

is achieved by proving $\lim_{x \to 2} (x+3) = 5$.

$\delta = \varepsilon$

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.

(a) $\boxed{\text{What radius produces such a disk?}}$

(b) If the machinist is allowed an error tolerance of $\pm 5\ cm^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? $\boxed{\text{What is the corresponding value of } \delta\text{?}}$

$A(r) = \pi r^2$

$\pi x^2 = f(x)$

x = radius (cm)

$f(x)$ = area (cm^2)

$\varepsilon = 5\ cm^2$

$L = 1000\ cm^2$

$a = \sqrt{\frac{1000}{\pi}}$

a. $\pi x^2 = 1000$

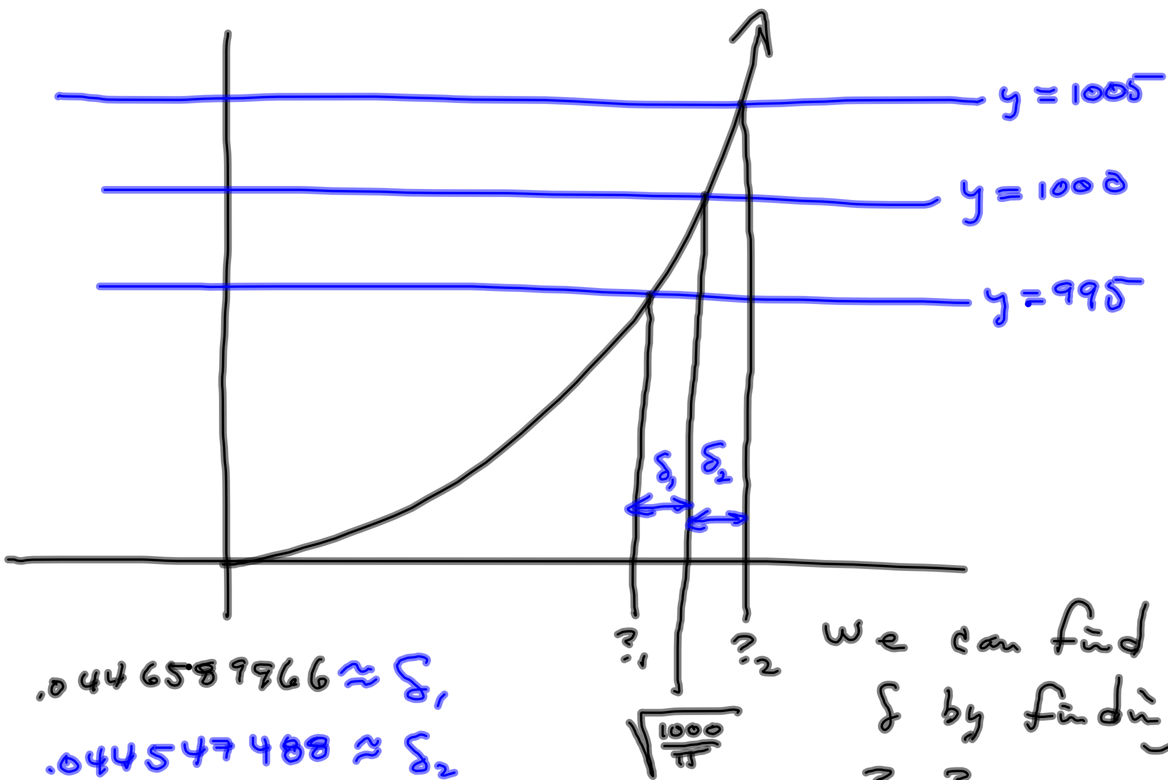
$x^2 = \frac{1000}{\pi}$

$|x| = \sqrt{\frac{1000}{\pi}}$ Take the positive:

$x = \sqrt{\frac{1000}{\pi}}$ cm

$\lim_{x \to \sqrt{\frac{1000}{\pi}}} \pi x^2 = 1000$

WINDOW:
xmin: 15
xmax: 20
xscl: 1
ymin: 700
ymax: 1300
yscl: 50



We can find δ by finding $?_1$, $?_2$.

$.0446589966 \approx \delta_1$

$.044547488 \approx \delta_2$

To be on the safe side make your tolerance on the Radius measure

on the small side, so make it something like 0.04 for your tolerance in the measure of the radius.

7. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

Your textbook thinks of this as a graphing calculator question. *I* think of it as an algebra question.

$\boxed{6}$ **DEFINITION** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

42. Prove, using Definition 6, that $\displaystyle \lim_{x \to -3} \frac{1}{(x + 3)^4} = \infty$.