2.3 CALCULATING LIMITS USING THE LIMIT LAWS

LIMIT LAWS Suppose that c is a constant and the limits

$$\boxed{\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)}$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ SUM LAW
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ DIFFERENCE LAW
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ CONSTANT MULTIPLE LAW
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ PRODUCT LAW
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ QUOTIENT LAW

$$\lim_{x\to 2}(x^2+7) = \lim_{x\to 2}(x^2) + \lim_{x\to 2}(7) = 2^2 + 7 = 11$$

SUM LAW 1. The limit of a sum is the sum of the limits.

DIFFERENCE LAW 2. The limit of a difference is the difference of the limits.

CONSTANT MULTIPLE 3. The limit of a constant times a function is the constant times the limit of the function.

PRODUCT LAW 4. The limit of a product is the product of the limits.

QUOTIENT LAW 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

Limit of the power is the power of the limit...

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

$$\lim_{x\to 2}(x^2) = \left(\lim_{x\to 2} x\right)^2 = (2)^2 = 4$$

With two more laws, below, we're ready to handle limits for polynomials, rational functions, powers and roots (if we stretch a point on what we know about rational powers as roots).

7. $\lim_{x\to a} c = c$ $\quad \lim_{x\to 2} 7 = 7$

8. $\lim_{x\to a} x = a$ $\quad \lim_{x\to 2} x = 2$

The following is a consequence of **6.** and **8.**

9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

$$\lim_{x\to 2} x^2 = \left(\lim_{x\to 2} x\right)^2 = 4$$

Since roots can be thought of as rational powers, this makes sense:

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer (If n is even, we assume that $a > 0$.)

$$\lim_{x\to 2} \sqrt[3]{x} = \sqrt[3]{\lim_{x\to 2}(x)}$$

Just as we would hope and expect:

ROOT LAW 11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

$$\lim_{x\to \frac{\pi}{6}} \sqrt[5]{\sin(x)} = \sqrt[5]{\lim_{x\to \frac{\pi}{6}} \sin(x)}$$

$$\lim_{x\to \frac{\pi}{6}} (\sin(x))^{\frac{1}{5}} = \left(\lim_{x\to \frac{\pi}{6}} \sin(x)\right)^{\frac{1}{5}}$$

It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference."

Putting more of the Laws together, we have this nice, tidy package:

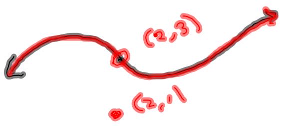
DIRECT SUBSTITUTION PROPERTY If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x \to 1} (x^2 - 2x + 1) = 1^2 - 2(1) + 1 = 0$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

This is irrespective of the actual values of $f(a)$ and $g(a)$!



$f(x)$, $g(x)$ agree, except at $x = 2$, where $g(x)$ has a hole.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 3$, even though $g(2) = 1$

$$\lim_{x \to -3} \frac{x^2 + 5x + 6}{x + 3} \neq \frac{\lim_{x \to -3} x^2 + 5x + 6}{\lim_{x \to -3} x + 3}, \text{ because}$$

$g(x) = x + 3 = 0$ @ $x = -3$, so quotient rule for limits doesn't hold up.

Here's how the pro's do 'em:

$$\frac{x^2 + 5x + 6}{x + 3} = \frac{\cancel{(x+3)}(x+2)}{\cancel{x+3}} = \underline{x+2} \quad (x \neq -3)$$

I can't let $x = -3$, but I CAN pass to the limit as x approaches -3, by last

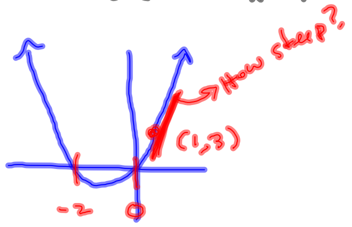
$$x + 2 \xrightarrow{x \to -3} -3 + 2 = -1 = \lim_{x \to -3} \frac{x^2 + 5x + 6}{x + 3}$$

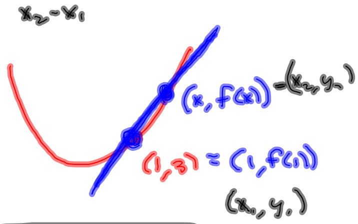
. Now, we have a way to find m_{tan}!

Find the slope of the tangent to

$x^2 + 2x$ at $x = 1$.

$$m_{sec} = \frac{f(x) - f(1)}{x - 1} = \frac{y_2 - y_1}{x_2 - x_1}$$





$$m_{tan} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \boxed{\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} =}$$

$$\lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} (x+3) = 1 + 3 = 4$$

The slope is EXACTLY 4 @ $x = 1$.

$$\left.\begin{aligned} f(x) &= x^2 + 2x \\ f'(x) &= 2x + 2 \end{aligned}\right\}$$

$$f'(1) = 2(1) + 2 = 4 = m_{tan}$$