Section 2.2 - The Limit of a Function

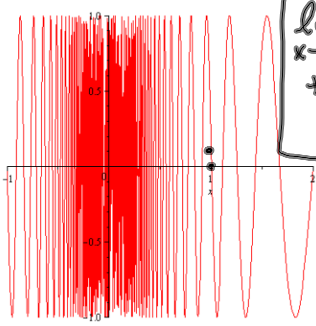
We've done some solid numerical investigations of limits - just take x closer and closer to the x-value of interest, and see if $f(x)$ appears to be approaching a single value. This didn't always work, for instance, when a function is oscillating very rapidly, like this one does:

$$f(x) = \sin\left(\frac{10\pi}{x}\right)$$

It turns out that the tangent to this f at $x = 1$ DOES exist, but approaching by 10ths is totally inadequate to the task. This sort of thing can happen ANY time you rely only on a numerical approach. That's why we are going to learn some rules that will guarantee the behavior of limits, so that we don't have to rely on mere numbers...

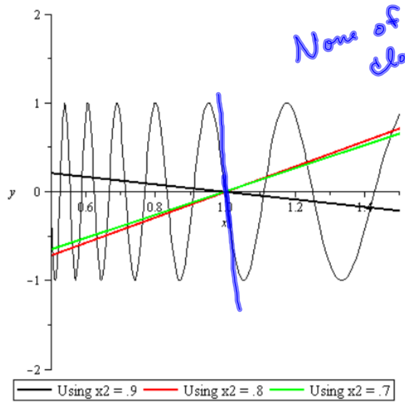
My *CAS* is lying to me on this one...

$f(x) = \sin\left(\frac{10\pi}{x}\right)$



$\boxed{\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \text{the SLOPE of } f(x) \text{ @ } x = 1.}$

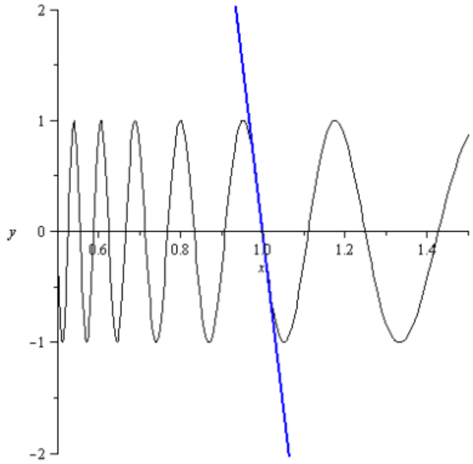
3 different secant lines, using $P = (1, 0)$, and points on the graph corresponding to $x = .7, .8, .9$:



None of 'em close to tangency @ $x=1$.

I THINK the actual $m_{tan} \approx -31.42$

I think we did this one and were asked to find the equation of the tangent line at $x = 1$. I don't recall getting -31.41592654 for its slope. Maybe it was -31.42 that I got the other day?



$$y = -10\pi(x-1) \approx -31.41592654x - 31.41592654$$

$\boxed{1}$ **DEFINITION** We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

$\boxed{2}$ **DEFINITION** We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

$\boxed{3}$ $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

An important point to remember:

If the limit exists, either the right- or left-hand limits will give its value; **However**,

if the limit does NOT exist, sometimes the right- or left-hand limits will trick you. Always be alert!

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

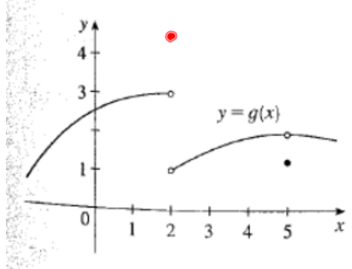


FIGURE 10

(a) $\lim_{x\to 2^-} g(x) = 3$

g gets close to 3 as x gets close to 2 (from the left)

(b) $\lim_{x\to 2^+} g(x) = 1$

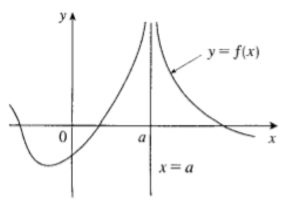
(c) $\lim_{x\to 2} g(x)$ $\nexists$

The left- & right-hand limits must agree. But this says NOTHING about g(2), itself.

4 DEFINITION Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.



I don't like these much.

FIGURE 12

Another notation for $\lim_{x\to a} f(x) = \infty$ is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Again the symbol ∞ is not a number, but the expression $\lim_{x\to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

$\lim_{x \to 0} \frac{1}{x^2} = \infty$ 1,000,000

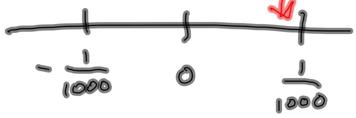
want $\frac{1}{x^2} > 1000000$

want $x^2 < \frac{1}{1000000}$

$|x| < \frac{1}{1000}$

If x is within $\frac{1}{1000}$ of 0, then $\frac{1}{x^2} > 1{,}000{,}000$

$\frac{1}{1001}$



$\frac{1}{\left(\frac{1}{1001}\right)^2} =$

2.1 Monday

2.2 Monday?

1.4 Thursday.