201 §6.5 #s 2, 5, 9, 10, 13, 20, 23

#s 1-8 Find $\overline{f}_{[a,b]}$.

(2) $f(x) = \sin(4x)$ on $[-\pi, \pi]$

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(4x)\,dx = \boxed{0 = \overline{f}}$$

ODD Func.

(5) $f(t) = t\sqrt{1+t^2}$ on $[0,5]$

$u = 1 + t^2 \Rightarrow du = 2t\,dt$, $t = 0 \Rightarrow u = 1$, $t = 5 \Rightarrow u = 26$

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{5}\int_0^5 t(1+t^2)^{\frac{1}{2}}\,dt$$

$\boxed{M1}$

$$= \frac{1}{5}\cdot\frac{1}{2}\int_0^5 (t^2+1)^{\frac{1}{2}}(2t\,dt) = \frac{1}{10}\int_1^{26} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{10}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{26} = \frac{1}{10}\left[\frac{2}{3}(26)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\cdot\frac{1}{10}\left[26\sqrt{26} - 1\right] = \boxed{\frac{1}{15}\left[26\sqrt{26} - 1\right]}$$

$\boxed{M2}$

So $\frac{1}{5}\int (t^2+1)^{\frac{1}{2}}\,dt = \frac{1}{5}\cdot\frac{1}{2}\int (t^2+1)^{\frac{1}{2}}(2t\,dt)$

$= \frac{1}{10}\left[\frac{2}{3}(t^2+1)^{\frac{3}{2}}\right] + C$, and we have

$$\frac{1}{10}\int_0^5 (t^2+1)^{\frac{1}{2}}(2t\,dt) = \frac{1}{15}\left[(t^2+1)^{\frac{3}{2}}\right]_0^5 = \frac{1}{15}\left[26^{\frac{3}{2}} - 1\right]$$

$$= \frac{1}{15}\left[\sqrt{26^3} - 1\right] = \frac{1}{15}\left[26\sqrt{26} - 1\right]$$

201 §6.5 #s 9, 10, 13, 20, 23

#s 9–12 (a) Find $\bar{f}_{[a,b]} = f_{ave}$

(b) Find $c \ni f(c) = \bar{f}_{[a,b]} = f_{ave}$

(c) Sketch f and ~~a~~ THE rectangle whose area is the same as the area under the graph of f.

9 $f(x) = (x-3)^2$ on $[2,5]$

(a) $f_{ave} = \frac{1}{3}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$

$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}\left[8+1\right] = \boxed{1 = f_{ave}} = \bar{f}_{[2,5]}$

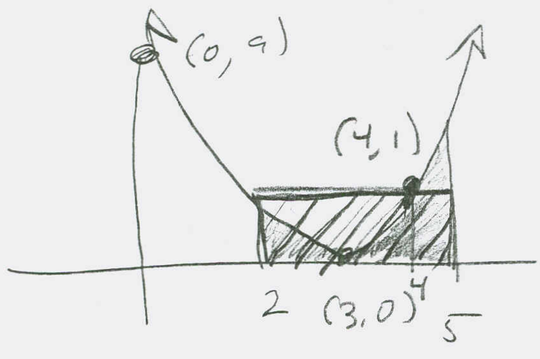
(b) $f(c) \overset{SET}{=} 1 \longrightarrow$

$(c-3)^2 = 1 \longrightarrow$

$c - 3 = \pm 1 \longrightarrow$

$c = 3 \pm 1$ → $c = 4$; $c = 2 \notin (2,5)$

(c)



Not the greatest sketch, but it will serve.

201 §6.5 #s 10, 13, 20, 23

(10) $f(x) = \sqrt{x}$ on $[0,4]$

(a) $\frac{1}{b-a}\int_a^b f(x)\,dx = f_{ave} = \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\,dx$

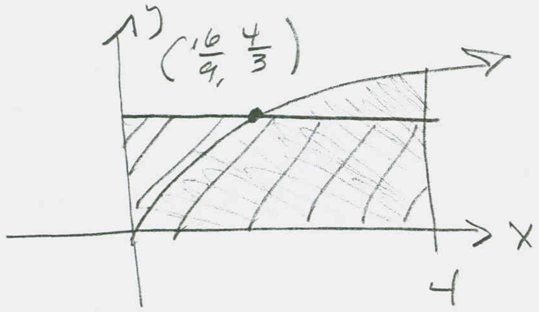
$= \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{1}{4}\cdot\frac{2}{3}\left[4^{\frac{3}{2}}\right] = \frac{1}{6}\left[2^3\right] = \frac{8}{6} = \boxed{\frac{4}{3} = f_{ave}}$

(b) $f(c) = f_{ave}$

$\sqrt{c} = \frac{4}{3}$

$\boxed{c = \frac{16}{9}}$

(c)



(13) If f is cnt's and $\int_1^3 f(x)\,dx = 8$, show that $f(x) = 4$ at least once in $[1,3]$.

Proof $\int_1^3 f(x)\,dx = 8 \Longrightarrow \frac{1}{3-1}\int_1^3 f(x)\,dx = \frac{1}{2}\cdot 8 = 4$

$\Longrightarrow \exists\, c \in (1,3) \ni f(c) = 4$, by MVT ∎

201 §6.5 #s 20, 23

(20) If a freely falling body starts from rest, then its displacement is $s = \frac{1}{2} g t^2$. Let velocity @ time T be v_T. Show that if we compute the Average of the velocities with respect to t, we get $v_{ave} = \frac{1}{2} v_T$, but if we compute the average of the velocities with respect to s, we get $v_{ave} = \frac{2}{3} v_T$!

<u>TOUGHIE</u> (conceptually)

(i) $v = v(s)$: $s = \frac{1}{2} g t^2 \Rightarrow \frac{ds}{dt} = v = gt$

And, since $s = \frac{1}{2} g t^2$, we have $\frac{2s}{g} = t^2 \Rightarrow$

$t = \sqrt{\frac{2s}{g}}$ $(t \geq 0)$. Replace t with this

expression: $v = g\sqrt{\frac{2s}{g}} = \sqrt{2sg}$

$$\therefore v_{ave} = \frac{1}{b-a} \int_a^b \sqrt{2sg}\, ds = \frac{1}{\frac{1}{2} g T^2} \int_0^{\frac{1}{2} g T^2} \sqrt{2g} \sqrt{s}\, ds$$

$s(0) = 0$, $s(T) = \frac{1}{2} g T^2$

$$= \frac{2}{gT^2} \cdot \frac{\sqrt{2g}}{1} \int_0^{\frac{1}{2} g T^2} s^{\frac{1}{2}} ds = \frac{2\sqrt{2}}{\sqrt{g}\, T^2} \left[\frac{2}{3} s^{\frac{3}{2}} \right]_0^{\frac{1}{2} g T^2}$$

$$= \frac{2\sqrt{2}}{\sqrt{g}\, T^2} \cdot \frac{2}{3} \left[\left(\frac{1}{2} g T^2 \right)^{\frac{3}{2}} \right] = \frac{4\sqrt{2}}{3\sqrt{g}\, T^2} \left[\frac{1}{2\sqrt{2}} g\sqrt{g}\, T^3 \right]$$

201 §6.5 #s 20, 23

20 cnt'd

$$= \frac{4\sqrt{2}}{3\sqrt{g}\,T^2} \cdot \frac{g\sqrt{g}\,T^3}{2\sqrt{2}} = \frac{2}{3} g T = \frac{2}{3} v_T \,!$$

(ii) $v = v(t) = gt \Longrightarrow$

$$v_{ave} = \frac{1}{T}\int_0^T g t \, dt = \frac{1}{T}\left[\frac{1}{2} g t^2\right]_0^T = \frac{1}{2} g T = \frac{1}{2} v_T \,!$$

This seems surprising, but in the 1st case, we looked at v as a function of how far the body had fallen; whereas, in the 2nd case, we looked at v more conventionally as a function of time.

23 Prove MVT for integrals by applying MVT for derivatives to the function $F(x) = \int_a^x f(t)\,dt$

GIVEN $f(t)$ is cnt's on $[a,b]$

CLAIM $\exists\ c \in (a,b) \ni f_{ave} = f(c)$.

Proof Let $x \in [a,b]$ and define $F(x) = \int_a^x f(t)\,dt$. Since f is cnt's, F is dif'b'l

201 §6.5 #23

23 cont'd Then by MVT for derivatives,

$\exists\ c \in (a,b) \ni F'(c) = \dfrac{F(b)-F(a)}{b-a}$. This

means that $f(c) = \dfrac{1}{b-a}\left[F(b)-F(a)\right]$

$$= \frac{1}{b-a}\int_a^b f(t)\,dt, \text{ by FTC II } \blacksquare$$