

3. The two numbers are x and $\frac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \frac{100}{x}$. $f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

8. We need to maximize P for $I \geq 0$. $P(I) = \frac{100I}{I^2 + I + 4} \Rightarrow$

$$P'(I) = \frac{(I^2 + I + 4)(100) - 100I(2I + 1)}{(I^2 + I + 4)^2} = \frac{100(I^2 + I + 4 - 2I^2 - I)}{(I^2 + I + 4)^2} = \frac{-100(I^2 - 4)}{(I^2 + I + 4)^2} = \frac{-100(I + 2)(I - 2)}{(I^2 + I + 4)^2}.$$

 $P'(I) > 0$ for $0 < I < 2$ and $P'(I) < 0$ for $I > 2$. Thus, P has an absolute maximum of $P(2) = 20$ at $I = 2$.

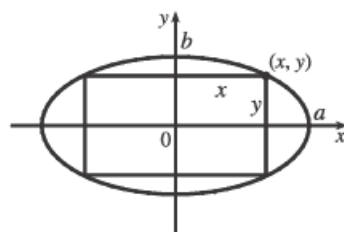
18. The square of the distance from a point (x, y) on the line $y = -6x + 9$ to the point $(-3, 1)$ is

$$D(x) = (x + 3)^2 + (y - 1)^2 = (x + 3)^2 + (-6x + 8)^2 = 37x^2 - 90x + 73. \quad D'(x) = 74x - 90, \text{ so } D'(x) = 0 \Leftrightarrow$$

$$x = \frac{45}{37}. \quad D''(x) = 74 > 0, \text{ so } D \text{ is concave upward for all } x. \text{ Thus, } D \text{ has an absolute minimum at } x = \frac{45}{37}. \text{ The point on the}$$

$$\text{line closest to } (-3, 1) \text{ is } \left(\frac{45}{37}, \frac{63}{37}\right).$$

22.



The area of the rectangle is $(2x)(2y) = 4xy$. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives

$$y = \frac{b}{a} \sqrt{a^2 - x^2}, \text{ so we maximize } A(x) = 4 \frac{b}{a} x \sqrt{a^2 - x^2}.$$

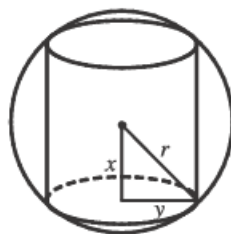
$$A'(x) = \frac{4b}{a} \left[x \cdot \frac{1}{2} (a^2 - x^2)^{-1/2} (-2x) + (a^2 - x^2)^{1/2} \cdot 1 \right]$$

$$= \frac{4b}{a} (a^2 - x^2)^{-1/2} [-x^2 + a^2 - x^2] = \frac{4b}{a \sqrt{a^2 - x^2}} [a^2 - 2x^2]$$

So the critical number is $x = \frac{1}{\sqrt{2}} a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}} b$, so the maximum area

$$\text{is } 4 \left(\frac{1}{\sqrt{2}} a \right) \left(\frac{1}{\sqrt{2}} b \right) = 2ab.$$

29.



The cylinder has surface area

$$\begin{aligned} 2(\text{area of the base}) + (\text{lateral surface area}) &= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) \\ &= 2\pi y^2 + 2\pi y(2x) \end{aligned}$$

 Now $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$, so the surface area is

$$\begin{aligned} S(x) &= 2\pi(r^2 - x^2) + 4\pi x \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r \\ &= 2\pi r^2 - 2\pi x^2 + 4\pi(x \sqrt{r^2 - x^2}) \end{aligned}$$

Thus,

$$\begin{aligned} S'(x) &= 0 - 4\pi x + 4\pi \left[x \cdot \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1 \right] \\ &= 4\pi \left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \right] = 4\pi \cdot \frac{-x \sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}} \end{aligned}$$

$$S'(x) = 0 \Rightarrow x \sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (*) \Rightarrow (x \sqrt{r^2 - x^2})^2 = (r^2 - 2x^2)^2 \Rightarrow$$

$$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \Rightarrow r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \Rightarrow 5x^4 - 5r^2x^2 + r^4 = 0.$$

 This is a quadratic equation in x^2 . By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10} r^2$, but we reject the root with the + sign since it

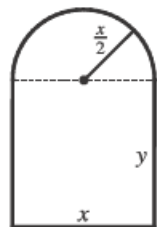
 doesn't satisfy (*). [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}} r$. Since $S(0) = S(r) = 0$, the

 maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10} r^2 \Rightarrow y^2 = r^2 - \frac{5 - \sqrt{5}}{10} r^2 = \frac{5 + \sqrt{5}}{10} r^2 \Rightarrow$

the surface area is

$$\begin{aligned} 2\pi \left(\frac{5 + \sqrt{5}}{10} \right) r^2 + 4\pi \sqrt{\frac{5 - \sqrt{5}}{10}} \sqrt{\frac{5 + \sqrt{5}}{10}} r^2 &= \pi r^2 \left[2 \cdot \frac{5 + \sqrt{5}}{10} + 4 \cdot \frac{\sqrt{(5 - \sqrt{5})(5 + \sqrt{5})}}{10} \right] = \pi r^2 \left[\frac{5 + \sqrt{5}}{5} + \frac{2\sqrt{20}}{5} \right] \\ &= \pi r^2 \left[\frac{5 + \sqrt{5} + 2 \cdot 2\sqrt{5}}{5} \right] = \pi r^2 \left[\frac{5 + 5\sqrt{5}}{5} \right] = \pi r^2 (1 + \sqrt{5}). \end{aligned}$$

30.



$$\text{Perimeter} = 30 \Rightarrow 2y + x + \pi \left(\frac{x}{2} \right) = 30 \Rightarrow$$

$$y = \frac{1}{2} \left(30 - x - \frac{\pi x}{2} \right) = 15 - \frac{x}{2} - \frac{\pi x}{4}.$$

 The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi \left(\frac{x}{2} \right)^2$, so $A(x) = x \left(15 - \frac{x}{2} - \frac{\pi x}{4} \right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

$$A'(x) = 15 - \left(1 + \frac{\pi}{4} \right) x = 0 \Rightarrow x = \frac{15}{1 + \pi/4} = \frac{60}{4 + \pi}. \quad A''(x) = -\left(1 + \frac{\pi}{4} \right) < 0, \text{ so this gives a maximum.}$$

The dimensions are $x = \frac{60}{4 + \pi}$ ft and $y = 15 - \frac{30}{4 + \pi} - \frac{15\pi}{4 + \pi} = \frac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \frac{30}{4 + \pi}$ ft, so the height of the rectangle is half the base.

56. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \Leftrightarrow y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.
- (b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \Rightarrow x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \13 .