201 §4.7 #s 3, 8, 18, 22, 29, 30, 56

(3) Find two positive #s whose product is 100 & whose sum is a minimum.

$$xy = 100$$

$S = \text{Sum} = x + y$ to be minimized

$$xy = 100 \Rightarrow y = \frac{100}{x} = 100x^{-1} \longrightarrow$$

$$S = x + 100x^{-1} \longrightarrow$$

$$\frac{dS}{dx} = 1 - 100x^{-2} = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2} \text{ SET } 0$$

$$\Rightarrow x = \pm 10 \longrightarrow \boxed{x = 10} \text{ is positive one} \longrightarrow$$

$$y = \frac{100}{x} = \frac{100}{10} = \boxed{10 = y}$$

(8) The rate in $\frac{\text{mg carbon}}{m^3 \cdot hr}$ of photosynthesis is $P = \frac{100I}{I^2 + I + 4}$, where I = light intensity (in thousands of foot-candles) For what Intensity is P a max?

$$\frac{dP}{dI} = \frac{100(I^2 + I + 4) - 100I(2I + 1)}{(I^2 + I + 4)^2}$$

$$= \frac{100I^2 + 100I + 400 - 200I^2 - 100I}{(I^2 + I + 4)^2} = \frac{-100I^2 + 400}{(I^2 + I + 4)^2}$$

$$\text{SET} = 0 \Rightarrow \boxed{I = 2 \text{ thousand foot-candles}}$$

201 §4.7 #s 18, 22, 29, 30, 56

(18) Find the point on the line $y + 6x = 9$ that is closest to $(-3, 1)$

$y = -6x + 9$

$D = \sqrt{(x-(-3))^2 + (y-1)^2} = \sqrt{(x+3)^2 + (-6x+9-1)^2}$

$= \sqrt{(x+3)^2 + (-6x+8)^2}$. To minimize D,

we minimize $D^2 = x^2 + 6x + 9 + 36x^2 - 96x + 64$

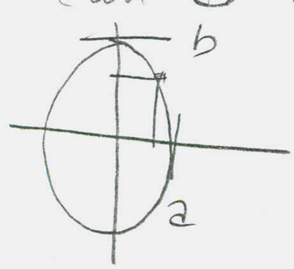
$= 37x^2 - 90x + 73 \Longrightarrow$

$\frac{dD^2}{dx} = 74x - 90 \overset{\text{SET}}{=} 0 \Longrightarrow x = \frac{90}{74}$

$\Rightarrow y = -6\left(\frac{90}{74}\right) + 9 = -3\left(\frac{90}{37}\right) + 9 = \frac{-270 + 333}{37}$

$= \frac{63}{37} \Longrightarrow$ $(x, y) = \left(\frac{90}{74}, \frac{63}{37}\right)$ minimizes the distance

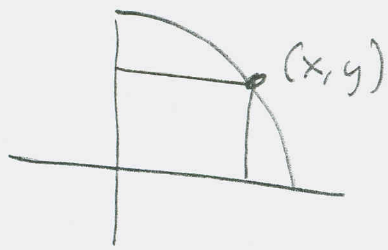
(22) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Use symmetry & zero-in on quadrant 1. At the end, multiply by 4.

201 § 4.7 #s 22, 29, 30, 56

22 Cnt'd



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$a^2y^2 = a^2b^2 - b^2x^2$$

$$y^2 = \frac{a^2b^2 - b^2x^2}{a^2} = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = xy = x \cdot \frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \frac{b}{a}x\,(a^2 - x^2)^{\frac{1}{2}} = A(x).$$

$$\Longrightarrow \frac{dA}{dx} = \frac{b}{a}\left[(a^2 - x^2)^{\frac{1}{2}} + x\left(\tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)\right)\right] \overset{\text{SET}}{=} 0$$

$$\Longrightarrow \frac{b}{a}\left[\frac{\sqrt{a^2 - x^2}}{1} \cdot \frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} - \frac{x^2}{\sqrt{a^2 - x^2}}\right] = 0 \longrightarrow$$

$$\frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}} = 0 \longrightarrow -2x^2 + a^2 = 0 \Longrightarrow 2x^2 = a^2 \Longrightarrow$$

$$x = \frac{1}{\sqrt{2}}a \Longrightarrow y = \frac{b}{a}\sqrt{a^2 - \left(\tfrac{1}{\sqrt{2}}a\right)^2}$$

$$= \frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{a}\sqrt{\frac{a^2}{2}} = \frac{ba}{a}\sqrt{\frac{1}{2}}$$

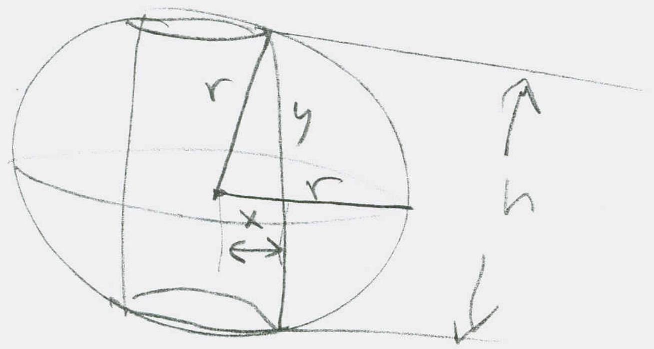
$$= b\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}b \Longrightarrow xy = \frac{\sqrt{2}}{2}a\,\frac{\sqrt{2}}{2}b = \frac{2}{4}ab$$

$$= \frac{1}{2}ab \longrightarrow$$

$$4xy = \boxed{2ab = \text{Area}}$$

201 §4.7 #s 29, 30, 56

29 Find largest SURFACE AREA of right circular cylinder inscribed in a sphere of radius r.



$x^2+y^2=r^2 \Longrightarrow y=\sqrt{r^2-x^2}$

and $y=\frac{h}{2} \Longrightarrow h=2y$

$A = \text{Area} = 2\cdot\pi x^2 + 2\pi x\cdot h = 2\pi x^2+4\pi x\sqrt{r^2-x^2}$

$\Longrightarrow \frac{dA}{dx} = 4\pi x + 4\pi\sqrt{r^2-x^2} + 4\pi x\cdot \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)$

SET 0 $\Longrightarrow$ $x+\sqrt{r^2-x^2} - \frac{x^2}{\sqrt{r^2-x^2}} = 0 \longrightarrow$

$\frac{x\sqrt{r^2-x^2} + r^2-x^2-x^2}{\sqrt{r^2-x^2}} = 0 \longrightarrow$

$-2x^2 + x\sqrt{r^2-x^2} + r^2 = 0$ RADICAL EQN.

$x\sqrt{r^2-x^2} = 2x^2-r^2$ SQUARE BOTH

$x^2(r^2-x^2) = 4x^4-4r^2x^2+r^4$

201 §4.7 #s 29, 30, 56

29 cnt'd

$$r^2x^2 - x^4 = 4x^4 - 4r^2x^2 + r^4$$

$$5x^4 - 5r^2x^2 + r^4 = 0$$

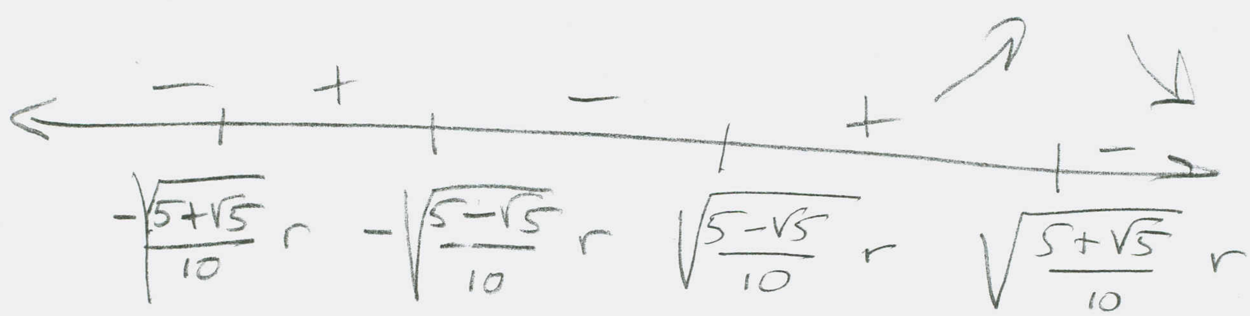
Let $u = x^2 \longrightarrow$

$$5u^2 - 5r^2u + r^4 = 0$$

$$b^2 - 4ac = (5r^2)^2 - 4(5)(r^4)$$
$$= 25r^4 - 20r^4 = 5r^4$$

$$u = \frac{5r^2 \pm \sqrt{5r^4}}{2(5)} = \frac{5r^2 \pm \sqrt{5}\, r^2}{10}$$

$$\Longrightarrow x = \pm\sqrt{\frac{5r^2 \pm \sqrt{5}\, r^2}{10}} = \pm\sqrt{\frac{5 \pm \sqrt{5}}{10}}\, r$$



Test values or analysis of sign of $\frac{dA}{dx}$ would give us the +/− pattern, above.

Strong algebra skills help, big-time.

Max is @ $x = \sqrt{\frac{5+\sqrt{5}}{10}}\, r$

$$y = \sqrt{r^2 - x^2} = \sqrt{r^2 - \left(\sqrt{\frac{5+\sqrt{5}}{10}}\, r\right)^2} = \sqrt{r^2 - \frac{5+\sqrt{5}}{10} r^2}$$

201 §4.7 #s 29, 30, 56

(29) cont'd

$$= \sqrt{\frac{10r^2 - 5r^2 - \sqrt{5}\,r^2}{10}} = \sqrt{\frac{5-\sqrt{5}}{10}}\,r$$

$\Rightarrow$ Something I didn't need, since area is already in terms of x.

From $y = \sqrt{r^2 - x^2}$ work

$$A = 2\pi\left(\sqrt{\frac{5+\sqrt{5}}{10}}\,r\right)^2 + 4\pi\sqrt{\frac{5+\sqrt{5}}{10}}\,r\sqrt{\frac{5-\sqrt{5}}{10}}\,r$$

$$= 2\pi\cdot\frac{5+\sqrt{5}}{10}r^2 + 4\pi\sqrt{\frac{25-5}{100}}\,r^2$$

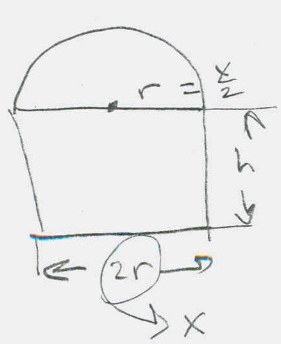
$$= 2\pi\,\frac{5+\sqrt{5}}{10}r^2 + 4\pi\,\frac{\sqrt{20}}{10}r^2$$

$$= \pi r^2\left[\frac{5+\sqrt{5}}{5} + \frac{2\cdot 2\sqrt{5}}{5}\right] = \left(\frac{5+\sqrt{5}+4\sqrt{5}}{5}\right)\pi r^2$$

$$= \left(\frac{5+5\sqrt{5}}{5}\right)\pi r^2 = \boxed{(1+\sqrt{5})\pi r^2}$$

Wow!

(30) Perimeter of window is 30 ft.



$$\text{Area} = \tfrac{1}{2}\pi\cdot\left(\tfrac{x}{2}\right)^2 + xh$$

$$\text{Perimeter} = x + 2h + \pi\left(\tfrac{x}{2}\right) = 30$$

$$x + 2h + \frac{\pi x}{2} = 30$$

$$\left(\frac{\pi}{2}+1\right)x + 2h = 30$$

$$2h = 30 - \left(\frac{\pi}{2}+1\right)x$$

$$h = 15 - \left(\frac{\pi}{4}+\frac{1}{2}\right)x$$

201 §4.7 #s 30, 56

30 cnt'd

$\Rightarrow$ Area $= A(x) = \frac{\pi}{8}x^2 + x\left(15 - \frac{\pi}{4}x - \frac{1}{2}x\right)$

$= \frac{\pi}{8}x^2 + 15x - \frac{\pi}{4}x^2 - \frac{1}{2}x^2 = -\frac{\pi}{8}x^2 - \frac{1}{2}x^2 + 15x$

$\Rightarrow A'(x) = -\frac{\pi}{4}x - x + 15$ SET $\equiv 0$

$\left(-\frac{\pi}{4} - 1\right)(x) = -15$

$\left(\frac{\pi}{4} + 1\right)x = 15$

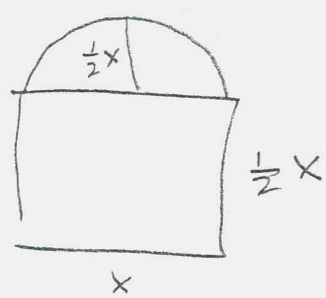
$\boxed{x = \dfrac{15}{\frac{\pi}{4}+1}} = \dfrac{15}{\frac{\pi+4}{4}} = \boxed{\dfrac{60}{\pi+4} = x}$

$\therefore h = 15 - \left(\frac{\pi}{4} + \frac{1}{2}\right)\left(\dfrac{15}{\frac{\pi}{4}+1}\right)$

$= 15 - \left(\dfrac{\pi+2}{4}\right)\left(\dfrac{15}{\frac{\pi+4}{4}}\right) = 15 - \left(\dfrac{\pi+2}{4}\right)\left(\dfrac{15(4)}{\pi+4}\right)$

$= \dfrac{15(\pi+4) - (\pi+2)(15)}{\pi+4} = \dfrac{15\pi + 60 - 15\pi - 30}{\pi+4}$

$= \boxed{\dfrac{30}{\pi+4}} = h = \frac{1}{2}$ the width of the window



201 §4.7 #56

(56) 20 necklaces per day @ \$10 per necklace.
when price is \$11, only 18 are sold.

(a) Find Demand = D(x) = # sold per day as a function of x = price (in \$). Assume D(x) is linear.

$$(x_1, D_1) = (10, 20)$$
$$(x_2, D_2) = (11, 18)$$
$$\Rightarrow m = \frac{18-20}{11-10} = \frac{-2}{1} = -2$$

$$D = m(x - x_1) + D_1$$
$$= -2(x-10) + 20$$
$$= -2x + 20 + 20$$
$$= -2x + 40$$

(b) If each necklace costs \$6, what price x will maximize his profit?

P(x) = Profit = R(x) − C(x) = Revenue − Cost

R(x) = (# sold) times (Price)

C(x) = (\$6/necklace) times (# of necklaces)

$$\therefore P(x) = D(x)\cdot x - 6\cdot D(x)$$
$$= (-2x+40)x - 6(-2x+40)$$
$$= -2x^2 + 40x + 12x - 80$$
$$= -2x^2 + 52x - 80$$

201 §4.7 #56

56 $P'(x) = -4x + 52 \overset{\text{SET}}{=} 0 \longrightarrow$ $\boxed{x = \$13 \text{ will maximize}}$ profit.

(Hidden analysis: $P(x)$ is parabola opening down. Its max is where $P'(x)=0$.)

other notes:

Raising the price to \$11 changed revenue to $(18)(11) = \$198$ and even though he sold fewer, he still brought in more money than selling 20 @ \$10 per necklace.