201 §4.6 #s 2, 5, 12, 22

#s 1-8 Graph!

2) $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

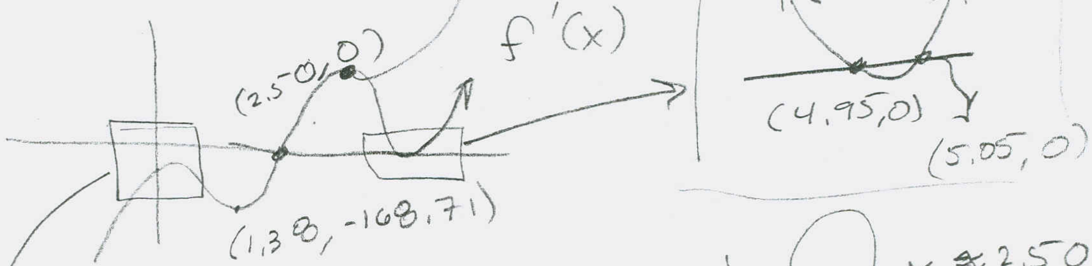
$= x(x^5 - 15x^4 + 75x^3 - 125x^2 - 1)$

$f'(x) = 6x^5 - 75x^4 + 300x^3 - 375x^2 - 1$

$[-5, 10] \times [-200, 200]$

will be a zero of f''!

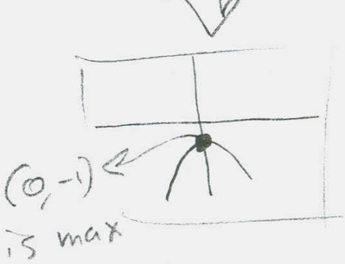
A graph of f'



So, a min @ $x \approx 2.50$

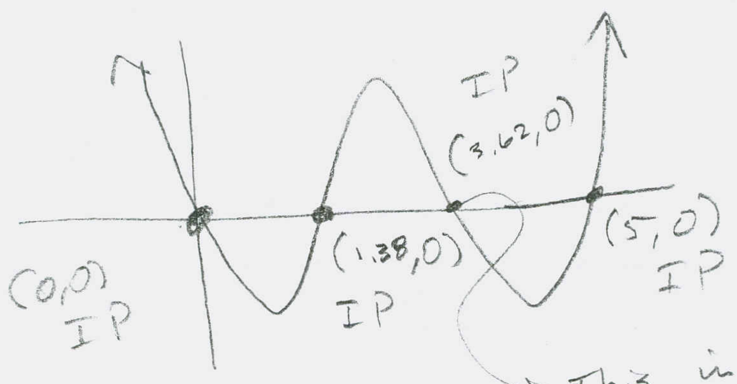
and $x \approx 5.05$

A max @ $x \approx 4.95$



A graph of f'':

$f''(x) = 30x^4 - 300x^3 + 900x^2 - 750x$



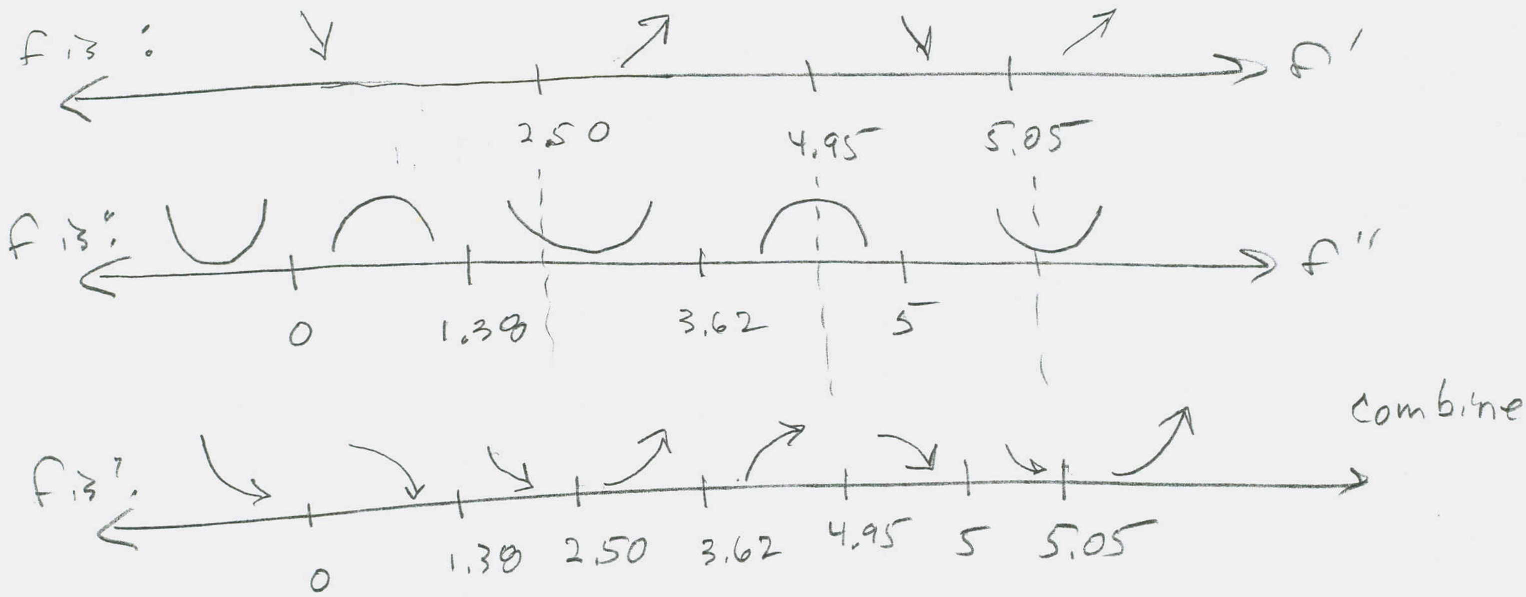
IP's @

$x = 0, 5$ &

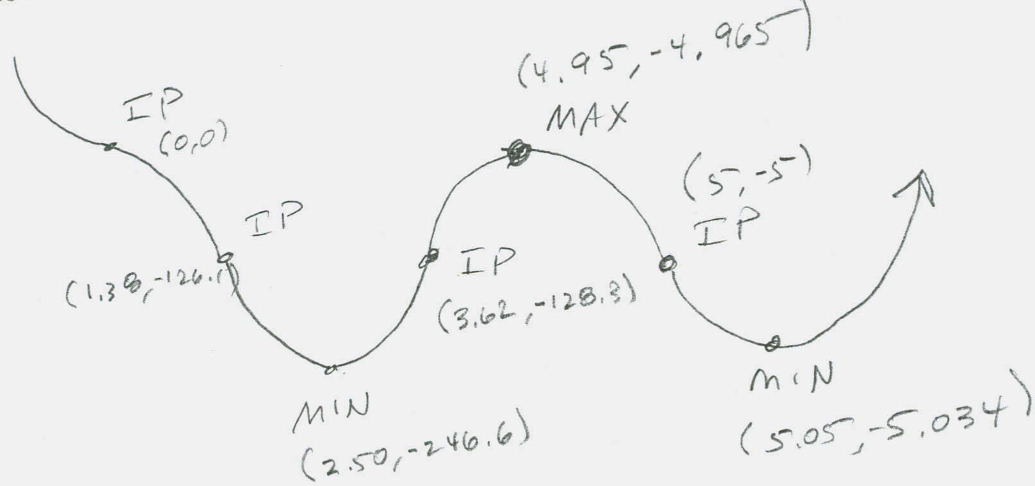
$x \approx 1.38, 3.62$

This intercept corresponds to the high point of f'

201 §4.6 #s 2,5,12,22



Rough sketch:



x	y	
0	0	IP
1.38	-126.1	IP
2.50	-246.6	MIN
3.62	-128.3	IP
4.95	-4.965	MAX
5	-5	IP
5.05	-5.034	MIN

Notice a few things:
The IP @ $x = 3.62$ needs to be WAY below the one @ $x = 5$!

201 §4.6 #s 2, 5, 12, 22

#2

A = (0,0) IP

B ≈ (1.38, −126.1) IP

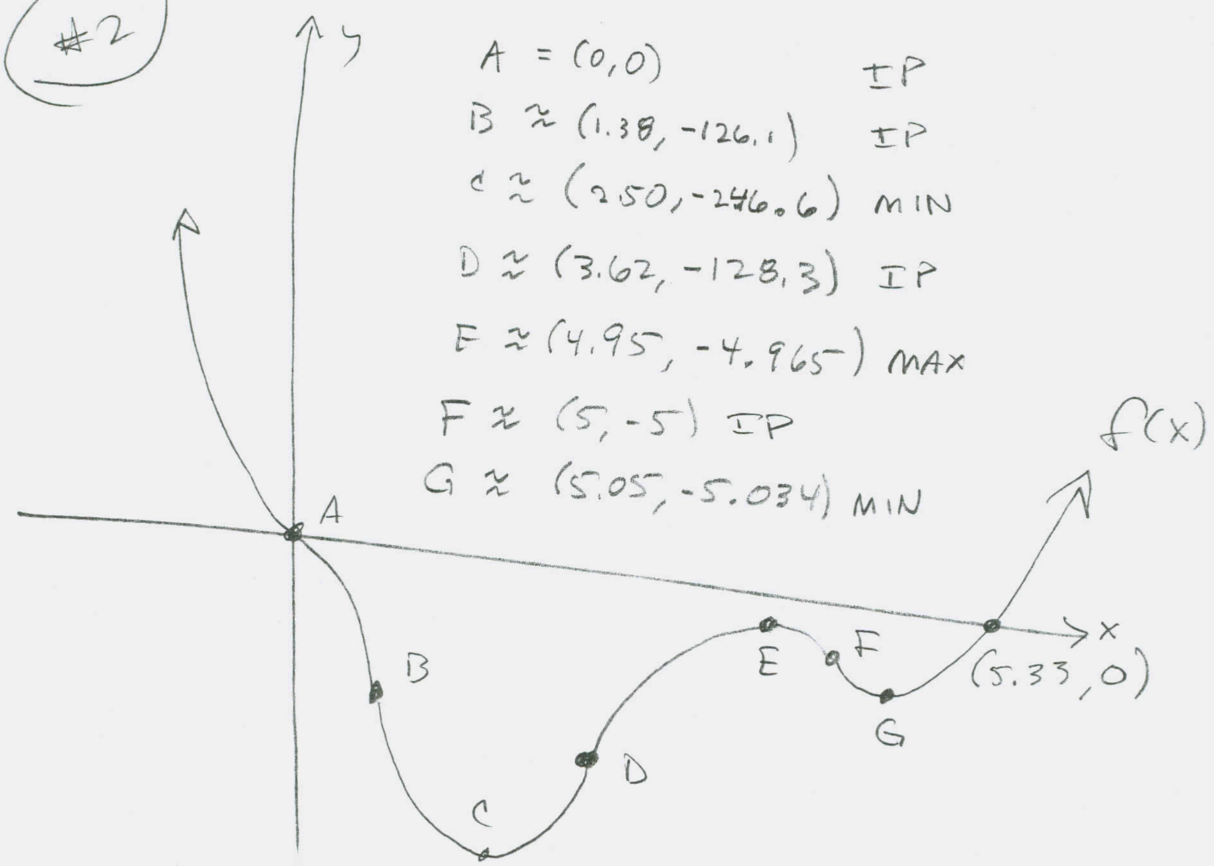
C ≈ (2.50, −246.6) MIN

D ≈ (3.62, −128.3) IP

E ≈ (4.95, −4.965) MAX

F ≈ (5, −5) IP

G ≈ (5.05, −5.034) MIN



Decreasing: $(-\infty, 2.50) \cup (4.95, 5.05)$

Increasing: $(2.50, 4.95) \cup (5.05, \infty)$

Concave up: $(-\infty, 0) \cup (1.38, 3.62) \cup (5, \infty)$

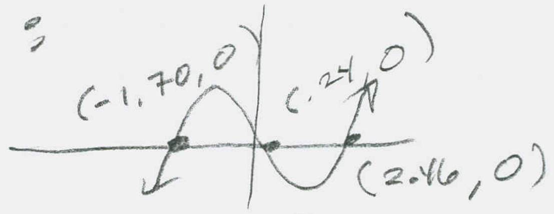
Concave down: $(0, 1.38) \cup (3.62, 5)$

201 §4.6 #s 5, 12, 22

5) $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1}$

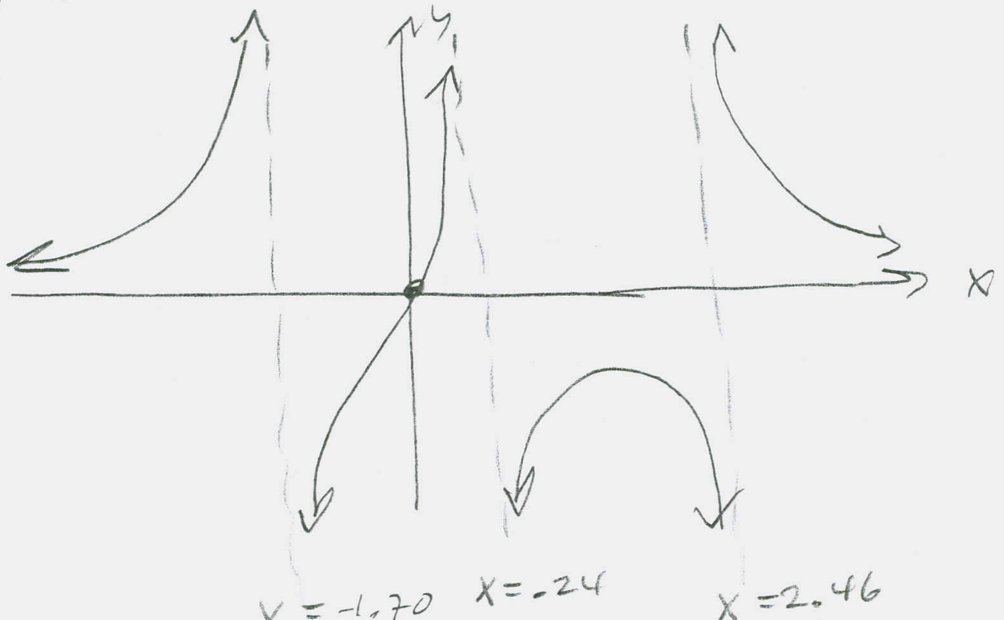
To get a handle on this, I graphed the denominator, just to get a feel for the vertical asymptotes

$x^3 - x^2 - 4x + 1$:



So $f(x)$ will have V.A. @ $x = -1.70$, $x = .24$, and $x = 2.46$.

$f(x) = 0 \Rightarrow x = 0 \leadsto (0,0)$



So this is a rough sketch of $f(x)$.

We don't know what its max/min values might be, nor where it has IPs, even though we know there's one between $x = -1.7$ & $x = +.24$

201 § 4.6 #s 5, 12, 22

5 cnt'd

$$f'(x) = \frac{(x^3 - x^2 - 4x + 1) - x(3x^2 - 2x - 4)}{(x^3 - x^2 - 4x + 1)^2}$$

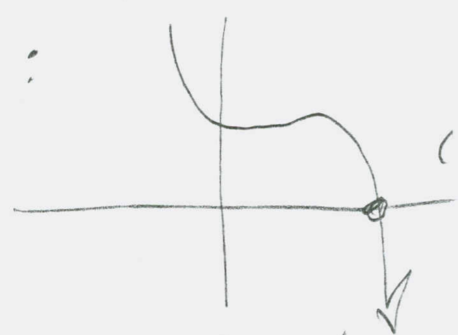
$$= \frac{x^3 - x^2 - 4x + 1 - 3x^3 + 2x^2 + 4x}{(x^3 - x^2 - 4x + 1)^2}$$

$$= \frac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2}$$

So f' also has V.A. @ same spots. But we want to find any real zeros f' might have.

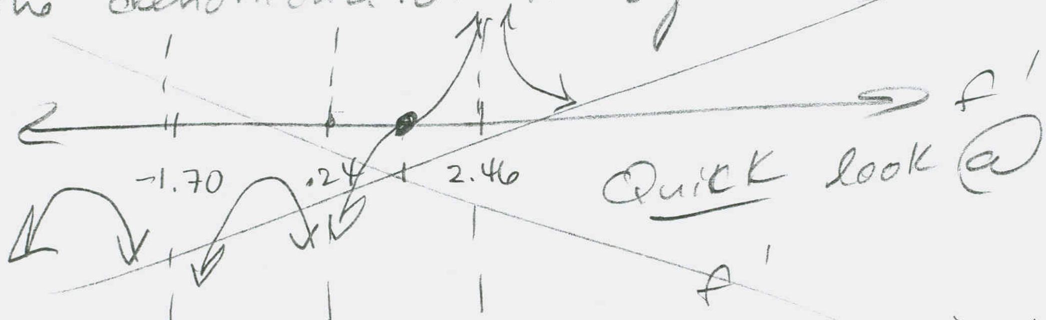
For this, I just graphed its numerator:

$-2x^3 + x^2 + 1$:



(1,0) is its only x-intercept

Note that zeros in the denominator of f' won't contribute any sign changes, because the denominator is squared.



Quick look @

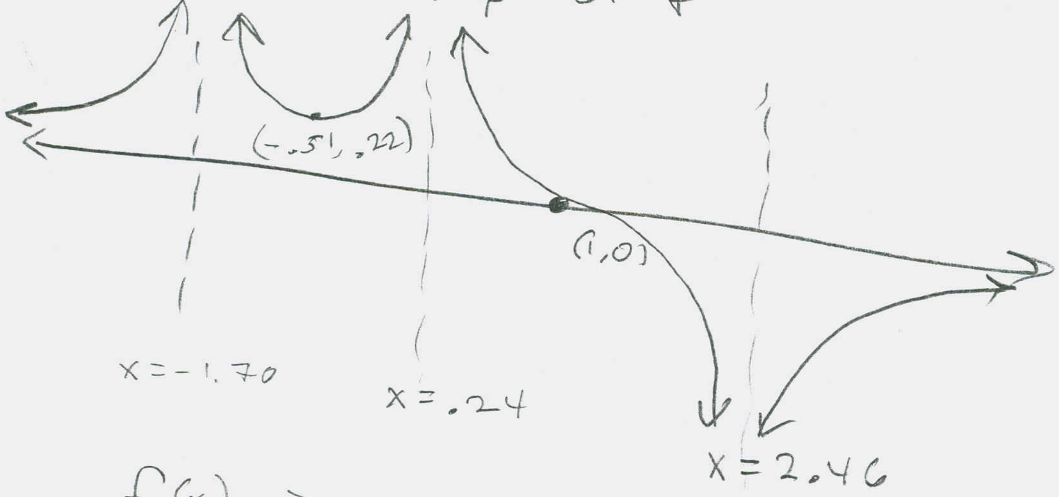
Nope. This is wrong.

I got the end behavior wrong. The $-2x^3$ makes $f'(x)$ negative to the right

201 §4.8 #s 5, 12, 22

5 cont'd

Graph of f'



$f(x)$ is

Increasing on $(-\infty, -1.70) \cup (-1.70, .24) \cup (.24, 1)$

Decreasing on $(1, 2.46) \cup (2.46, \infty)$

The minimum values for $f'(x)$ in the above graph will correspond to x-intercepts for $f''(x)$, hence they represent candidates for inflection points.

My calculator says $(-.506, .21766)$ is IP (A min for f', a root for f'')

$f''(x) = [(-6x^2+2x)(x^3-x^2-4x+1) - (-2x^3+x^2+1)(2(x^3-x^2-4x+1)(3x^2-2x-4)] / (x^3-x^2-4x+1)^4$ we can cancel a factor of x^3-x^2-4x+1:

$$f''(x) = \frac{-6x^2+2x - (-2x^3+x^2+1)(2)(3x^2-2x-4)}{(x^3-x^2-4x+1)^3}$$

f'' has same asymptotes as f', only the 3rd power means signs will change.

201 §4.6 #s 5, 12, 22

5 cnt'd

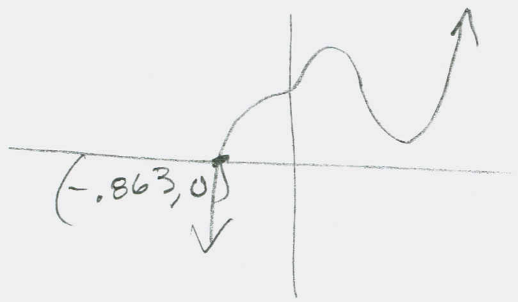
$$f''(x) = \frac{-6x^2+2x-(-12x^5+14x^4+12x^3-2x^2-4x-8)}{(x^3-x^2-4x+1)^3}$$

Scratch: $(-4x^3+2x^2+2)(3x^2-2x-4) =$

$$\begin{array}{rrrrrr} -12x^5 & +8x^4 & +16x^3 & & & \\ & +6x^4 & -4x^3 & -8x^2 & & \\ & & & +6x^2 & -4x & -8 \\ \hline -12x^5 & +14x^4 & +12x^3 & -2x^2 & -4x & -8 \end{array}$$

$$f''(x) = \frac{12x^5-14x^4-12x^3+4x^2+6x+8}{(x^3-x^2-4x+1)^3}$$

Same V.A. Real zeros from looking @ numerator:



The extra wiggle near the y-axis accounts for 2 potential extrema

The reason I bring this up is because we're looking @ $12x^5$ + smaller, so that means we've use up all 4 possible turning points, even if the wiggle is only a terrace point! That means we don't have to look for any more x-intercepts outside the window.

201 §4.6 #s 5, 12, 22.

<u>NOTE</u> The zero of f'' was NOT the same as the one predicted. So, either I messed up $f'(x)$ or, more likely, I messed-up $f''(x)$, because $f''(x)$ is messier. Trying again on f'' (or comparing with a colleague or resorting to technology):

$$f'(x) = \frac{-2x^3+x^2+1}{(x^3-x^2-4x+1)^2} \Longrightarrow$$

THIS was what I missed!

$$f''(x) = \frac{(-6x^2+2x)(x^3-x^2-4x+1)^2 - (-2x^3+x^2+1)(2(x^3-x^2-4x+1)(3x^2-2x-4)}{(x^3-x^2-4x+1)^4}$$

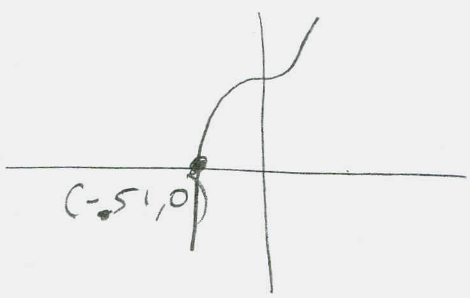
$= \ldots$ we can factor x^3-x^2-4x+1 out of the numerator and cancel it with one from the denominator ...

$$= \frac{(-6x^2+2x)(x^3-x^2-4x+1) - (-2x^3+x^2+1)(3x^2-2x-4)(2)}{(x^3-x^2-4x+1)^3}$$

$$= \ldots = \frac{2(3x^5-3x^4+5x^3-6x^2+3x+4)}{(x^3-x^2-4x+1)^3}$$

Same vertical asymptotes.

Graph of the numerator (looking for zeros of f''):



$x \approx -.5061189$ is zero.

This DOES agree with the min we found for $f'(x)$

GOOD SIGN.

MAPLE ASSIST & CHECK of WORK.

$with(plots):$

$f := x \to \frac{x}{(x^3 - x^2 - 4 \cdot x + 1)}$

$$x \to \frac{x}{x^3 - x^2 - 4x + 1} \tag{1}$$

$fp := D(f)$

$$x \to \frac{1}{x^3 - x^2 - 4x + 1} - \frac{x(3x^2 - 2x - 4)}{(x^3 - x^2 - 4x + 1)^2} \tag{2}$$

$expand(fp(x))$

$$\frac{1}{x^3 - x^2 - 4x + 1} - \frac{3x^3}{(x^3 - x^2 - 4x + 1)^2} + \frac{2x^2}{(x^3 - x^2 - 4x + 1)^2} + \frac{4x}{(x^3 - x^2 - 4x + 1)^2} \tag{3}$$

$normal(\%)$

$$-\frac{2x^3 - x^2 - 1}{(x^3 - x^2 - 4x + 1)^2} = f'(x) \tag{4}$$

$fpp := D(fp)$

$$x \to -\frac{2(3x^2 - 2x - 4)}{(x^3 - x^2 - 4x + 1)^2} + \frac{2x(3x^2 - 2x - 4)^2}{(x^3 - x^2 - 4x + 1)^3} - \frac{x(6x - 2)}{(x^3 - x^2 - 4x + 1)^2} \tag{5}$$

$expand(fpp(x))$

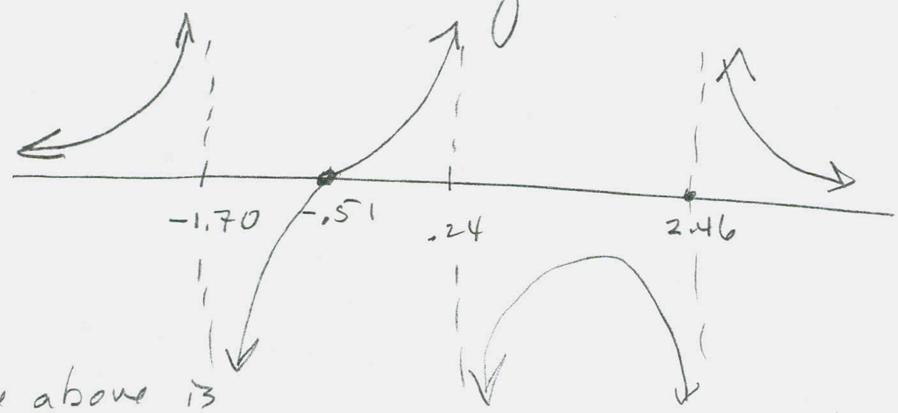
$$-\frac{12x^2}{(x^3 - x^2 - 4x + 1)^2} + \frac{6x}{(x^3 - x^2 - 4x + 1)^2} + \frac{8}{(x^3 - x^2 - 4x + 1)^2} \tag{6}$$
$$+ \frac{18x^5}{(x^3 - x^2 - 4x + 1)^3} - \frac{24x^4}{(x^3 - x^2 - 4x + 1)^3} - \frac{40x^3}{(x^3 - x^2 - 4x + 1)^3}$$
$$+ \frac{32x^2}{(x^3 - x^2 - 4x + 1)^3} + \frac{32x}{(x^3 - x^2 - 4x + 1)^3}$$

$normal(\%)$

$$\frac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3} = f''(x) \tag{7}$$

201 §4.6 #s 5, 12, 22

(5) Cnt'd Now for quick sketch of f'':



The above is based on sign changing at each boundary.

Test values

$f''(-2) \approx 7.78$ + ✓

$f''(-.7) \approx -.1524$ − ✓

$f''(-.3) \approx .26482$ + ✓

$f''(1) \approx -.222$ − ✓

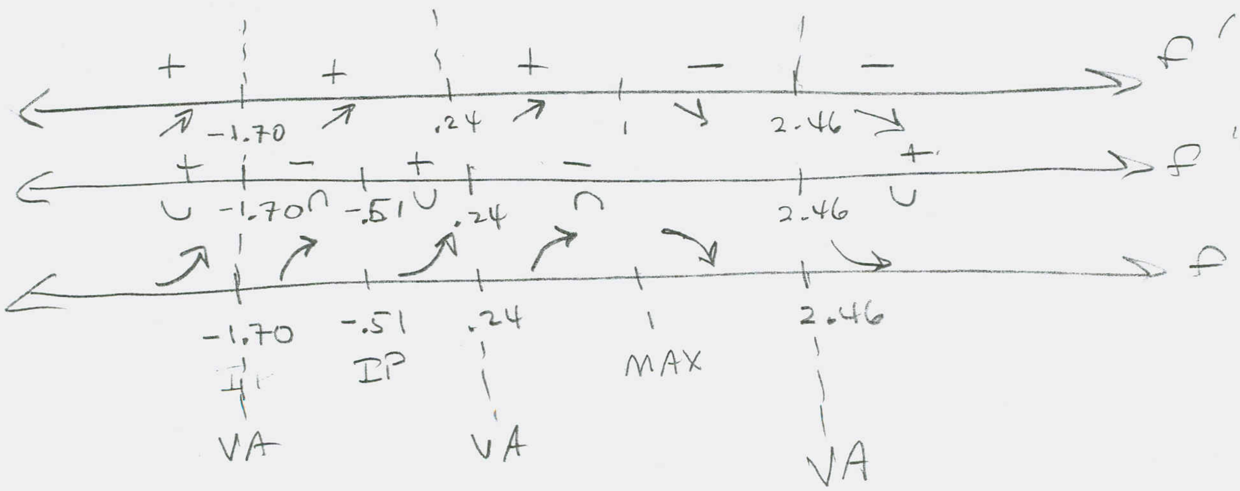
$f''(3) \approx 1.691$ + ✓

This is also in agreement with graphing calculator result for $f''(x)$.

Concavity:

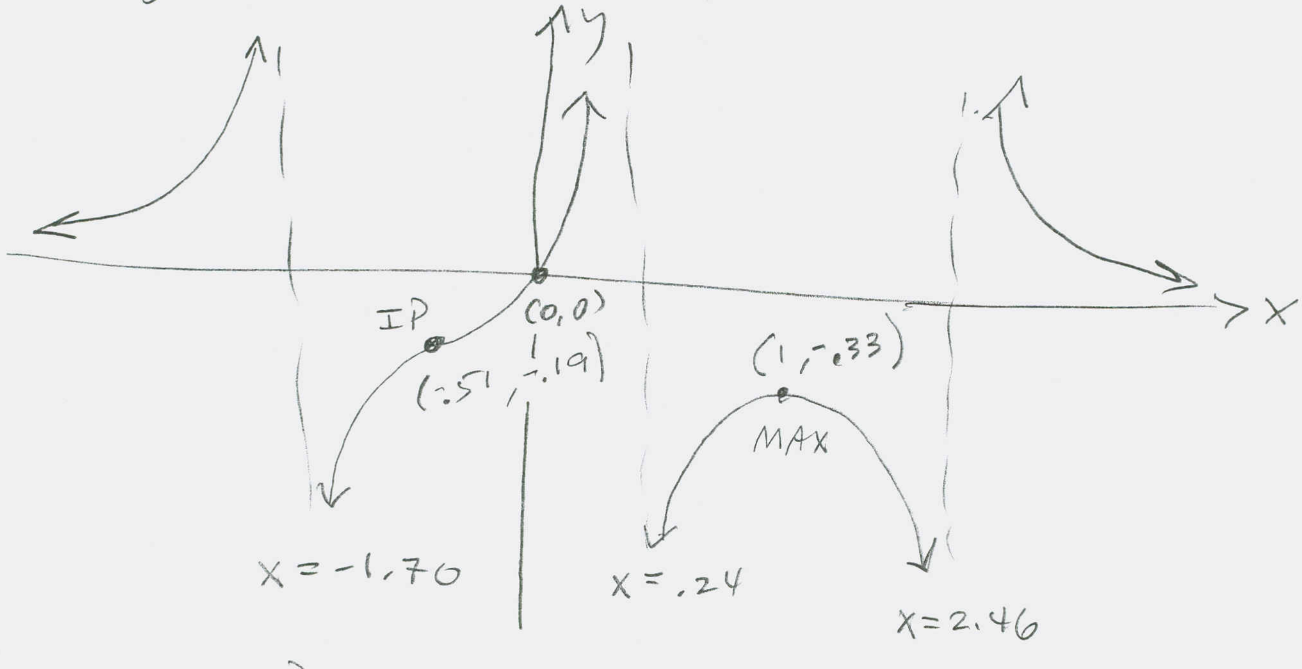
up: $(-\infty, -1.70) \cup (-.51, .24) \cup (2.46, \infty)$

down: $(-1.70, -.51) \cup (.24, 2.46)$



201 §4.6 #s 5, 12, 22

Rough sketch of f:



$f(-.5061189) \approx -.1918$

$f(1) \approx -.3333$

Max confirmed by graphing calculator.

Heck, that's good enough!

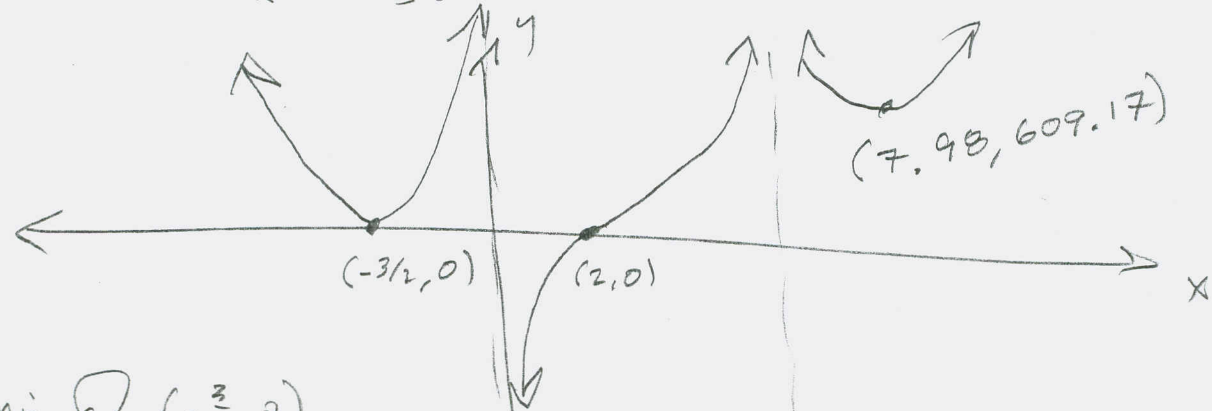
The ONE thing our grapher couldn't tell us without finding derivatives was the inflection point at $(-.5061189, -.1918)$ (approx).

We STILL used the grapher to locate $x \approx -.5061189$ by finding min of $f'(x)$ & confirming with finding zero of $f''(x)$.

201 §4.6 #s 12, 22

(12) We give a hand sketch & then use a graphing device to actually find max/min points

$$f(x) = \frac{(2x+3)^2 (x-2)^5}{x^3 (x-5)^2} \qquad \frac{(2x)^2 (x)^5}{(x^3)(x^2)} = \frac{2^2 x^7}{x^5} = 2^2 x^2$$



min @ $(-\frac{3}{2}, 0)$

Grapher says

min @ (7.98, 609.17) (approx)

$x \approx 7.9799087$

$y \approx 609.17403$

(22) $f(x) = x\sqrt{c^2 - x^2}$ Draw some versions of $f(x)$ for different values of c. Talk about what you see.

$\mathcal{D}(f) = [-c, c]$

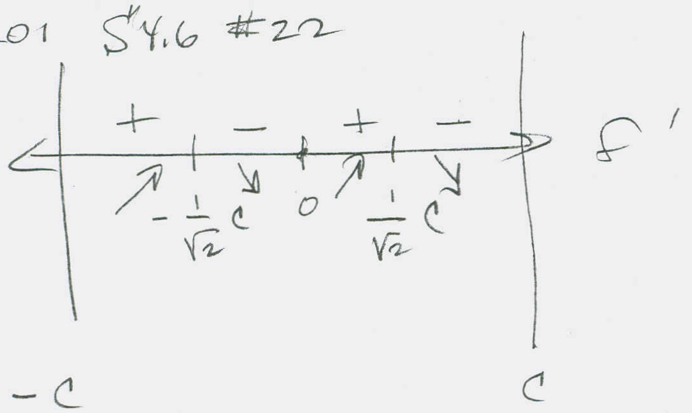
$$f'(x) = \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}} \stackrel{\text{SET}}{=} 0 \Longrightarrow 2x^2 = c^2 \Longrightarrow$$

$x = \pm \frac{1}{\sqrt{2}} c$ is where max/min vals lie.

$$f''(x) = \frac{x(2x^2 - 3c^2)}{(c^2 - x^2)^{3/2}} \stackrel{\text{SET}}{=} 0 \longrightarrow x = 0, \pm\sqrt{\tfrac{3}{2}}\, c$$

But $\pm\sqrt{\frac{3}{2}}\, c \notin \mathcal{D}$.

201 §4.6 #22



All the IP candidates are outside the domain, except for the one corresponding to $x=0$.