MAT 201 FORMAL PROOF OF THE CLAIM in §4.4 #62.

Claim: $\lim\limits_{x \to \infty} \frac{\sqrt{4x^2+1}}{x+1} = 2$

Scratch: Use what you want to find the N you need $\left|\frac{\sqrt{4x^2+1}}{x+1} - 2\right| < \varepsilon$. We can always make the left hand side BIGGER & then clamp a lid on that bigger thing: First, some algebra.

$$\left|\frac{\sqrt{4x^2+1}}{x+1} - \frac{2(x+1)}{x+1}\right| = \left|\frac{\sqrt{4x^2+1} - 2(x+1)}{x+1}\right|$$

$$= \left|\frac{\left(\sqrt{4x^2+1} - 2(x+1)\right)\left(\sqrt{4x^2+1} + 2(x+1)\right)}{(x+1)\left(\sqrt{4x^2+1} + 2(x+1)\right)}\right|$$ Rationalize

$$\left|\frac{4x^2+1 - (2(x+1))^2}{(x+1)\left(\sqrt{4x^2+1} + 2(x+1)\right)}\right| = \left|\frac{4x^2+1-4x^2-8x-4}{(x+1)\left(\sqrt{4x^2+1}+2(x+1)\right)}\right|$$ Simplify

$$\left|\frac{-8x-3}{(x+1)\left(\sqrt{4x^2+1}+2(x+1)\right)}\right| = \left|\frac{8x+3}{(x+1)\left(\sqrt{4x^2+1}+2(x+1)\right)}\right|$$ Property of absolute value

$x \to +\infty \Rightarrow x > 3$

Just to cancel with the "2"

$$\left|\frac{8x+x}{(x)\left(\sqrt{4x^2}\right)}\right| = \left|\frac{9x}{2x^2}\right| = \left|\frac{9}{2x}\right| < \left|\frac{10}{2x}\right| = \frac{5}{x}$$

ow $\frac{5}{x} < \varepsilon \iff \frac{5}{\varepsilon} < x$. So let $N = \max\{4, \frac{5}{\varepsilon}\}$. Now

rite the proof:

$\boxed{\text{ROOF:}}$ Let $\varepsilon > 0$ be given. Define $N = \max\{4, \frac{5}{\varepsilon}\}$.

want it bigger than 3, anyway.

en if $x > N$, we have

$\left|\frac{\sqrt{4x^2+1}}{x+1} - 2\right| = \ldots$ Algebra from scratch $\ldots < \left|\frac{8x+x}{(x)(2x)}\right|$

$$\left|\frac{9x}{2x^2}\right| < \frac{10}{2x} = \frac{5}{x} \le \frac{5}{\left(\frac{5}{\varepsilon}\right)} = \varepsilon \quad \blacksquare$$

Probably could/should drop absolute values

The interplay of the pure proof with the concrete application:

Pure proof is challenging, but once it's done, you can use that $N = \frac{5}{\varepsilon}$ for ANY desired tolerance (ANY ε), so $\varepsilon = .01$, $\varepsilon = .001$ doesn't make the question any different.

People in the real world can just use $N = \frac{5}{\varepsilon}$ and get something workable in seconds.

On the other hand, the N we get in the proof is probably quite a bit bigger than we need. If the real-world application entails greater \$ for greater N, we probably want an estimate on N that's as small as possible, but will still work.

So something concrete <u>OR</u> a formal proof that is more careful about throwing stuff away might be the ideal.