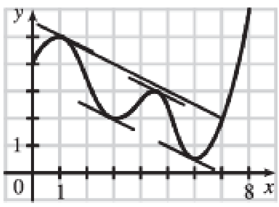
4. $f(x) = \cos 2x,\ [\pi/8, 7\pi/8]$. f, being the composite of the cosine function and the polynomial $2x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[\pi/8, 7\pi/8]$ and differentiable on $(\pi/8, 7\pi/8)$. Also, $f\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2} = f\left(\frac{7\pi}{8}\right)$. $f'(c) = 0 \quad \Leftrightarrow \quad -2\sin 2c = 0 \quad \Leftrightarrow \quad \sin 2c = 0 \quad \Leftrightarrow \quad 2c = \pi n \quad \Leftrightarrow \quad c = \frac{\pi}{2}n$. If $n = 1$, then $c = \frac{\pi}{2}$, which is in the open interval $(\pi/8, 7\pi/8)$, so $c = \frac{\pi}{2}$ satisfies the conclusion of Rolle's Theorem.

6. $f(x) = \tan x$. $f(0) = \tan 0 = 0 = \tan \pi = f(\pi)$. $f'(x) = \sec^2 x \geq 1$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'\left(\frac{\pi}{2}\right)$ does not exist, and so f is not differentiable on $(0, \pi)$. (Also, $f(x)$ is not continuous on $[0, \pi]$.)

8. $\dfrac{f(7) - f(1)}{7 - 1} = \dfrac{2 - 5}{6} = -\dfrac{1}{2}$. The values of c which satisfy $f'(c) = -\frac{1}{2}$ seem to be about $c = 1.1, 2.8, 4.6$, and 5.8.



12. $f(x) = x^3 + x - 1,\ [0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} \quad \Leftrightarrow$

$3c^2 + 1 = \dfrac{9 - (-1)}{2} \quad \Leftrightarrow \quad 3c^2 = 5 - 1 \quad \Leftrightarrow \quad c^2 = \frac{4}{3} \quad \Leftrightarrow \quad c = \pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0, 2)$.

13. $f(x) = \sqrt[3]{x},\ [0, 1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad \dfrac{1}{3c^{2/3}} = \dfrac{f(1) - f(0)}{1 - 0} \quad \Leftrightarrow \quad \dfrac{1}{3c^{2/3}} = \dfrac{1 - 0}{1} \quad \Leftrightarrow$

$3c^{2/3} = 1 \quad \Leftrightarrow \quad c^{2/3} = \frac{1}{3} \quad \Leftrightarrow \quad c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \quad \Leftrightarrow \quad c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0, 1)$.

14. $f(x) = \dfrac{x}{x + 2},\ [1, 4]$. f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow$

$\dfrac{2}{(c + 2)^2} = \dfrac{\frac{2}{3} - \frac{1}{3}}{4 - 1} \quad \Leftrightarrow \quad (c + 2)^2 = 18 \quad \Leftrightarrow \quad c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1, 4)$.

20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)((x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4-1)$ for some $c \in (1,4)$. But for every $c \in (1,4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4-1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

24. If $3 \leq f'(x) \leq 5$ for all x, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8-2)$ for some c in $[2,8]$. (f is differentiable for all x, so, in particular, f is differentiable on $(2,8)$ and continuous on $[2,8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that

$6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \quad \Rightarrow \quad 18 \leq f(8) - f(2) \leq 30.$

26. Let $h = f - g$. Then since f and g are continuous on $[a,b]$ and differentiable on (a,b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b-a)$. Since $h'(c) < 0$, $h'(c)(b-a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b,a]$ and differentiable on (b,a). By the Mean Value Theorem, there is a number $c \in (b,a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a-b) = (\cos c)(a-b)$. Thus, $|\sin a - \sin b| \leq |\cos c|\,|b-a| \leq |a-b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \leq |b-a| = |a-b|$. If $a = b$, both sides of the inequality are 0.

34. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a,b]$, we find that there is a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$. But then $f'(c) = \dfrac{b-a}{b-a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.