201 §4.2 #s 4, 6, 8, 12, 13, 14, 20, 23, 24, 26, 29, 34

#s 1-4 Verify that the hypotheses of Rolle's Theorem are satisfied. Then find all numbers c that satisfy the conclusion of Rolle's.

(4) $f(x) = 3x^2 - 12x + 5$ on $[1, 3]$

f is cnt's and difb'l $\forall x \in \mathbb{R}$, so it's cnts on $[1,3]$ ✓ and difb'l on $(1,3)$ ✓

$$\left.\begin{aligned} f(1) &= 3(1)^2 - 12(1) + 5 = 3 - 12 + 5 = -4 \\ f(3) &= 3(3)^2 - 12(3) + 5 = 27 - 36 + 5 = -4 \end{aligned}\right\} ✓$$

$f'(x) = 6x - 12 \overset{\text{SET}}{=} 0 \implies$

$6x = 12$

$x = \boxed{2 = c}$ is value of x that satisfies the conclusion $\exists\, c \in (1,3) \ni f'(c) = 0$.

(6) Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$, but $\nexists\, c \in (0, \pi) \ni f'(c) = 0$. Why does this not contradict Rolle's Thm?

$f(0) = \tan(0) = 0 = \tan(\pi)$ ✓

$f'(x) = \sec^2 x \overset{\text{SET}}{=} 0 \implies$

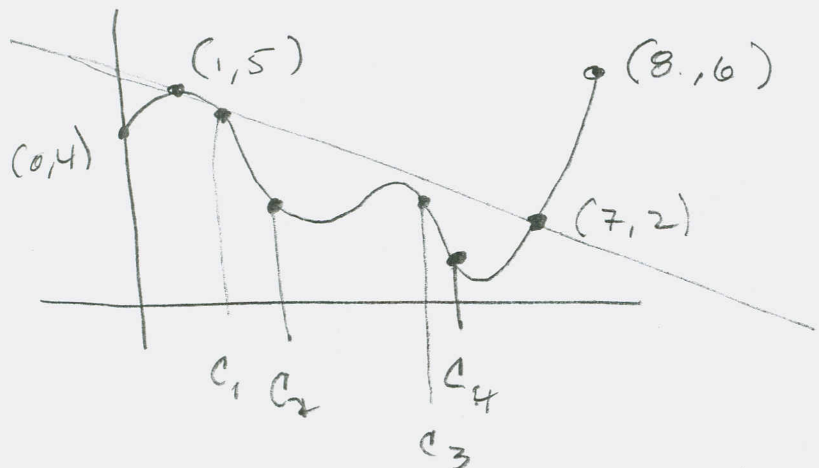
$\frac{1}{\cos^2 x} = 0 \implies$

$1 = 0$ ✗.

This is no contradiction, because the hypotheses of Rolle's aren't satisfied on $[0, \pi]$: f isn't cnt's @ $x = \frac{\pi}{2}$.

201 § 4.2 #s 8, 12, 13, 14, 20, 23, 24, 26, 29, 34

(8) Use the graph in #7 to estimate the values of c that satisfy the MVT conclusion on $[1, 7]$



$m = \frac{2-5}{7-1} = \frac{-3}{6} = -\frac{1}{2}$

It looks like there are 4 values of c that satisfy MVT conclusion. My sketch is poor, but using the better (book) sketch, it looks like $\boxed{c = 1.5, 2.5, 4.8, \& 5.7}$ are pretty close to where $f'(x) = -\frac{1}{2}$.

#s 11-14 Verify the function satisfies MVT hypotheses on $[a, b]$. Then find all values c that satisfy MVT conclusion.

(12) $f(x) = x^3 + x - 1$ on $[0, 2]$

f is cnt^{s} on $\mathbb{R} \supset [0,2]$

f is difb^{l} on $\mathbb{R} \supset (0,2)$

201 §4.2 #s 12-14, 20, 23, 24, 26, 29, 34

(12) cont'd

$f(0) = 0^3 + 0 - 1 = -1 \rightsquigarrow (0,-1)$

$f(2) = 2^3 + 2 - 1 = 9 \rightsquigarrow (2,9)$

$m = \dfrac{9+1}{2-0} = 5$

$f'(x) = x^3 + x - 1 \overset{\text{SET}}{=} 5$ → NOT $f'(x)$

$\Longrightarrow x^3 + x - 6 = 0$

$\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & 1 & -6 \\ & & 3 & 9 & \\ \hline & 1 & 3 & 10 & \end{array}$$

$$\begin{array}{r|rrrr} -3 & 1 & 0 & 1 & -6 \\ & & -3 & 9 & \\ \hline & 1 & -3 & 10 & \end{array}$$

$$\begin{array}{r|rrrr} 6 & 1 & 0 & 1 & -6 \\ & & 6 & 36 & \\ \hline & 1 & 6 & & \end{array}$$

$$\begin{array}{r|rrrr} -6 & 1 & 0 & 1 & -6 \\ & & -6 & 36 & \\ \hline & 1 & -6 & & \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 1 & -6 \\ & & 1 & 1 & \\ \hline & 1 & 1 & 2 & \text{NO} \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 1 & -6 \\ & & -1 & 1 & \\ \hline & 1 & -1 & & \text{NO} \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -6 \\ & & 2 & 4 & \\ \hline & 1 & 2 & 5 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 1 & -6 \\ & & -2 & 4 & \\ \hline & 1 & -2 & 5 & \end{array}$$

$f'(x) = 3x^2 + 1 \overset{\text{SET}}{=} 5 \Longrightarrow \boxed{c = \dfrac{2\sqrt{3}}{3}}$ $\checkmark$

$3x^2 = 4$

$x^2 = \dfrac{4}{3}$

$x = \pm \dfrac{2}{\sqrt{3}} = \pm \dfrac{2\sqrt{3}}{3}$

where $f'(x) = m_{AVG} = 5$

201 § 4.2 #s 13, 14, 20, 23, 24, 26, 29, 34

(13) $f(x) = \sqrt[3]{x}$ on $[0,1]$

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is cnt^s on $\mathbb{R} \supset [0,1]$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x}}$ exist^s on $(\mathbb{R} \setminus \{0\}) \supset (0,1)$

$f(0) = 0 \rightsquigarrow (0,0)$

$f(1) = 1 \rightsquigarrow (1,1)$ $\quad m_{AVG} = 1 \implies$

$f'(x) = \dfrac{1}{3\sqrt[3]{x}} \overset{SET}{=} 1 \implies 1 = 3\sqrt[3]{x} \implies$

$\sqrt[3]{x} = \frac{1}{3} \implies x = \left(\frac{1}{3}\right)^3 = \boxed{\frac{1}{27} = c}$

(14) $f(x) = \dfrac{x}{x+2}$ on $[1,4]$

$\implies f'(x) = \dfrac{1(x+2) - x(1)}{(x+2)^2} = \dfrac{x+2-x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

f is cnt^s on $(\mathbb{R} \setminus \{-2\}) \supset [1,4]$ and

f is difb^l on $(\mathbb{R} \setminus \{-2\}) \supset (1,4)$

$f(1) = \dfrac{1}{1+2} = \dfrac{1}{3} \rightsquigarrow (1, \frac{1}{3})$

$\implies m_{AVG} = \dfrac{\frac{2}{3} - \frac{1}{3}}{4-1} = \dfrac{\frac{1}{3}}{3} = \dfrac{1}{9}$

$f(4) = \dfrac{4}{4+2} = \dfrac{2}{3} \rightsquigarrow (4, \frac{2}{3})$

$f'(x) = \dfrac{2}{(x+2)^2} \overset{SET}{=} \dfrac{1}{9}$

$\implies (x+2)^2 = 18 \implies x+2 = \pm\sqrt{18} = \pm 3\sqrt{2}$

$\implies \boxed{c = -2 + 3\sqrt{3}}$

201 §4.2 #s 20, 23, 24, 26, 29, 34

(20) Show that $x^4 + 4x + C = 0$ has at <u>MOST</u> two real roots.

$x^4 + 4x + C = 0$ Let $C = 0$. Then

$x^4 + 4x = 0$

$x(x^3 + 4) = 0$ has Two Real Roots.

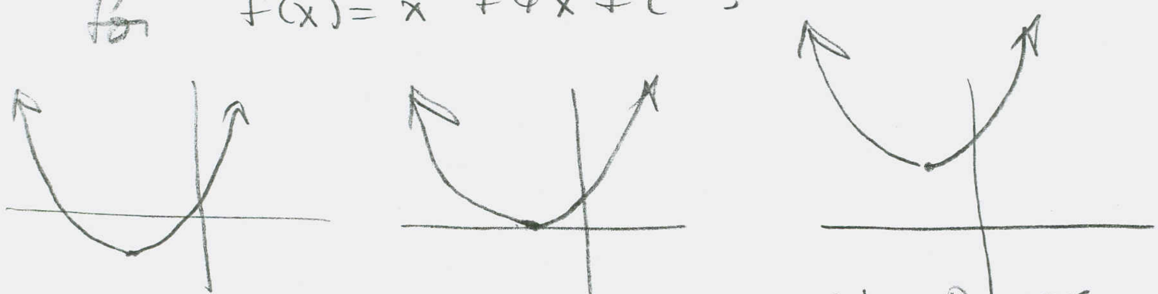
This doesn't help directly, but it gives some insight. Consider the derivative of $f(x) = x^4 + 4x + C$:

$f'(x) = 4x^3 + 4$. This is the key

$f'(x)$ changes sign only once, since

$4x^3 + 4 = 0 \implies x^3 = -1 \implies x = -1$ is

its only root. So here are the possibilities for $f(x) = x^4 + 4x + C$:



TWO ROOTS ONE ROOT. NO ROOTS.

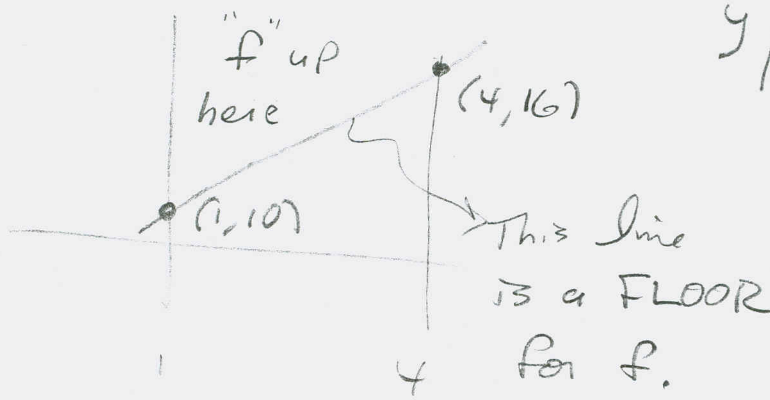
To have <u>more</u> real roots, you'd need a local max to bring $f(x)$ back down to meet the x-axis.

201 §4.2 #s 23, 24, 26, 29, 34

(23) If $f(1)=10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ be?

Consider the line containing $(1,10)$ and w/ slope $m=2$.

$y = 2(x-1)+10$. It's BELOW or EQUAL $f(x)$ on $[1,4]$

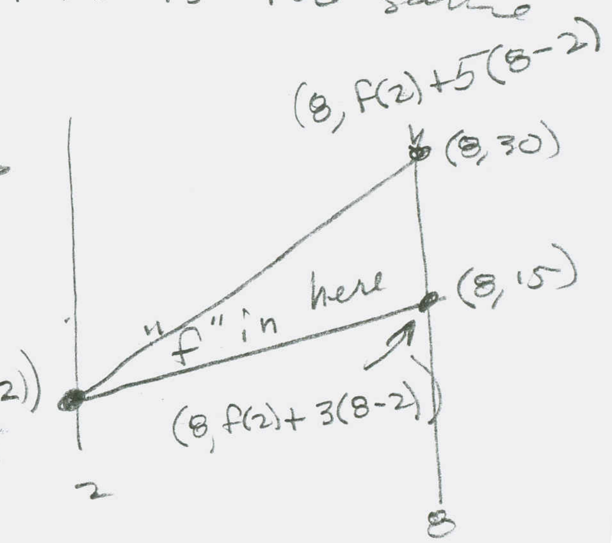


$$y\big|_{x=4} = 2(4-1)+10 = 2(3)+10 = 16$$

$\therefore$ $\boxed{f(4) \geq 16}$

(24) f $3 \leq f'(x) \leq 5$ $\forall x$. Then $18 \leq f(8)-f(2) \leq 30$. This is the same idea as before.

$$3 \leq \frac{f(8)-f(2)}{8-2} \leq 5 \Longrightarrow$$



$$18 \leq f(8)-f(2) \leq 30$$

Alternate:

$$f(2)+3(8-2) \leq f(8) \leq f(2)+5(8-2)$$

$$f(2)+18 \leq f(8) \leq f(2)+30$$

$$\Rightarrow 18 \leq f(8)-f(2) \leq 30.$$

201 §4.2 #s 26, 29, 34

(26) $\mathcal{S}$ f, g cnts on $[a,b]$ & difbl on (a,b)

$\mathcal{S}$ $f(a) = g(a)$ & $f'(x) < g'(x)$ $\forall$ $x \in (a,b)$.

Prove that $f(b) < g(b)$ (HINT MVT on $h = f - g$)

OK. Using the hint:

Let $h(x) = f(x) - g(x)$.

Then $h(a) = 0$, and $h(b) = f(b) - g(b)$.

We want to show, then, that $h(b) < 0$.

Let $m_{AVG} = \dfrac{h(b) - h(a)}{b - a}$. By MVT, $\exists$

$c \in (a,b)$ $\ni$ $h'(c) = m_{AVG}$

But $h'(c) = f'(c) - g'(c) < 0$, so

$$m_{AVG} = \frac{h(b) - h(a)}{b - a} = \frac{h(b)}{b-a} < 0.$$ Since

$b > a$, we know $b - a > 0$. This means

$h(b) = f(b) - g(b) < 0$, i.e., $f(b) < g(b)$ ▨

201 §4.2 #s 29, 34

(29) Use MVT to prove

$|\sin a - \sin b| \le |a-b| \quad \forall\ a \,\&\, b.$

Hmmmmmm.

$\dfrac{\sin a - \sin b}{a-b}$ looks like (is) the difference quotient for $\sin x$. We know that as $a \to b$, this will approach $\cos b$, and $|\cos b| \le 1$. Now, how to connect this to what we want, namely, that

$$\left| \frac{\sin a - \sin b}{a-b} \right| \le 1.$$

I see an indirect proof (by contradiction)

Proof

The trivial case: $a=b \Longrightarrow |\sin a - \sin b| = 0 = |a-b|$. Check. Now, suppose there is a pair of #s (a,b) such that

$|\sin(a) - \sin(b)| > |a-b|$ Then

$\left| \dfrac{\sin(a)-\sin(b)}{a-b} \right| > 1.$ By mean value theorem, then, there must be a $c \in (a,b)$ such that

$\dfrac{d}{dx}[\sin x] = \cos x$ is greater than 1.

This is a contradiction ∎

201 §4.2 #s 29, 34

29 OK. I've roughed it in. I think I have the key. Now for a nice writeup.

Claim

$|\sin a - \sin b| \leq |a-b|$ for any $a, b \in \mathbb{R}$.

If $a=b$, we're done, since $0 \leq 0$. Now, suppose (wlog) $a < b$. Then, for $\sin x$, m_{avg} on $[a,b]$ is

$m_{avg} = \dfrac{\sin a - \sin b}{a-b}$. Since $\sin x$ is cnts and difb^le everywhere, we apply MVT to

$f(x) = \sin x \Longrightarrow$

$f'(x) = \cos x \Longrightarrow$

There is a $c \in (a,b)$ such that

$$\frac{\sin a - \sin b}{a-b} = \cos(c), \text{ and so}$$

$$\left|\frac{\sin a - \sin b}{a-b}\right| = |\cos(c)| \leq 1, \text{ which}$$

implies $|\sin a - \sin b| \leq |a-b|$ ▨

34 Prove that if $f'(x) \neq 1\ \forall x \in \mathbb{R}$, then $f(x)$ has at MOST one fixed point.

Hmmmmmmm, Ṡ $a \neq b$ AND $f(a) = a$ & $f(b) = b$. That is, suppose that there are at least two fixed points. OK, I see it. This will lead to a $c \in (a,b)$ such that $f'(c) = 1$.

201 §4.2 #34

34 cnt'd. Now, we write the proof.

<u>Claim</u>

If $f'(x) \neq 1 \ \forall x \in \mathbb{R}$, then f has at most one fixed point.

<u>Proof</u>

§ f has <u>more</u> than one fixed point.
(NOT CONCLUSION)

Then $\exists\, a, b$ with $a \neq b$ and $f(a) = a$, $f(b) = b$.

Then (wlog, assume $a < b$) on $[a, b]$,

$$m_{AVG} = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$, and by MVT,

$\exists\, c \in (a, b) \ni f'(c) = m_{avg} = 1$. So, if f has <u>more</u> than one fixed point, it also must satisfy $f'(c) = 1$ for some c.
(NOT HYPO)

This completes the proof $\blacksquare$

We proved $A \Longrightarrow B$ by proving

$$\underbrace{\text{NOT } B \Longrightarrow \text{NOT } A}_{\text{The contrapositive of } A \longrightarrow B}$$

is logically equivalent to $A \Longrightarrow B$.