$\frac{d}{dx}\left[\sqrt{x}\ \sin x\right]$ by the definition

$f(x) = \sqrt{x}\ \sin x \implies$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}\ \sin(x+h) - (\sqrt{x}\ \sin x)}{h}$$

$$= \frac{\sqrt{x+h}\,(\sin x \cos h + \sin h \cos x) - \sqrt{x}\ \sin x}{h}$$

$$= \frac{\sqrt{x+h}\ \sin x \cos h - \sqrt{x+h}\ \sin h \cos x - \sqrt{x}\ \sin x}{h} =$$

Here's the trick: will vanish as $h \to 0$

$$\frac{\sqrt{x+h}\ \sin x \cos h - \sqrt{x+h}\ \sin x + \sqrt{x+h}\ \sin x - \sqrt{x+h}\ \sin h \cos x - \sqrt{x}\ \sin x}{h}$$

$$= \frac{(\sqrt{x+h} - \sqrt{x})\sin x}{h} - \frac{\sqrt{x+h}\ \cos x \sin h}{h}$$

stuck the $\sin x$ terms together

$$= (\sin x)\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right) - (\cos x)\left(\frac{\sqrt{x+h}\ \sin h}{h}\right)$$

Recognized the $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ living in there.

$$\xrightarrow{h \to 0} (\sin x)\cdot\frac{d}{dx}\left[\sqrt{x}\right] - (\cos x)(\sqrt{x}\cdot 1)$$

The $\frac{\sqrt{x+h}\ \sin h}{h}$ has a nice limit.

$$= (\sin x)\cdot\frac{1}{2\sqrt{x}} - (\cos x)\sqrt{x}$$

I still cheated by recognizing the $\frac{d}{dx}[\sqrt{x}]$ that pops out, but we've done $\sqrt{x}$ by the definition of derivative, already

For Catherine Cozad.

The trick was making the $\cos h$ go away by adding and subtracting something that would give a $\frac{\cos h - 1}{h} \xrightarrow{h \to 0} 0$ to get rid of it.