201 §3.3 II #5 70, 71, 73, 78, 84, 85, 89, 90, 91, 96, 97, 99, 100

(70) If f is difbl, find the derivative of each of the following:

(a) $y = x^2 f(x) \longrightarrow \boxed{y' = 2x f(x) + x^2 f'(x)}$

(b) $y = \frac{f(x)}{x^2} \Longrightarrow y' = \frac{f'(x)x^2 - (f(x))(2x)}{x^4}$

$$\boxed{y' = \frac{x f'(x) - f(x)}{x^3}}$$

(c) $y = \frac{x^2}{f(x)} \Longrightarrow \boxed{y' = \frac{2x f(x) - x^2 f'(x)}{f^2(x)}}$

(d) $y = \frac{1 + x f(x)}{\sqrt{x}} \Longrightarrow y' \cancel{= (f(x) + x f'(x))} \sqrt{x}$

$$\boxed{y' = \frac{(f(x) + x f'(x))\sqrt{x} - (1 + x f(x))\frac{1}{2}x^{-\frac{1}{2}}}{x}}$$

is fare enough

(71) Find points on the curve $f(x) = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

$f'(x) = 6x^2 + 6x - 12 \overset{\text{SET}}{=} 0 \longrightarrow$

$6(x^2 + x - 2) = 0 \longrightarrow 6(x+2)(x-1) = 0 \longrightarrow$

$x \in \{-2, 1\}$

201 $\S$ 3.3 II #s 71, 73, 78, 84, 85, 89, 90, 91, 96, 97, 99, ~~100~~

71 cnt'd

$f(-2)$:

$$\begin{array}{r|rrrr} -2 & 2 & 3 & -12 & 1 \\ & & -4 & 2 & 20 \\ \hline & 2 & -1 & -10 & 21 = f(-2) \end{array}$$

$f(1)$:

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -12 & 1 \\ & & 2 & 5 & -7 \\ \hline & 2 & 5 & -7 & -6 = f(1) \end{array}$$

The pts are $(-2, 21)$ & $(1, -6)$

73 Show that $y = f(x) = 6x^3 + 5x - 3$ has NO points where tangent slope is 4.

We show that $f'(x) = 4$ has no real solution:

$f'(x) = 18x^2 + 5 \overset{\text{SET}}{=} 4 \longrightarrow$

$18x^2 = -1$ which clearly has no real roots.

So, either $f'(x) > 4$ always or $f'(x) < 4$ always. Which is it?

201 §3.3 II #s 78, 84, 85, 89, 90, 91, 96, 97, 99, 100

(78) Where does the normal line to the parabola @ (1,0) intersect the parabola a 2nd time? $f(x) = x - x^2 \Longrightarrow$

$f'(x) = 1 - 2x \Longrightarrow$

$m_{tan}\Big|_{x=1} = f'(1) = 1 - 2 = -1 = m \longrightarrow$

$m_\perp = +1$. Normal line:

$y = 1(x-1) + 0 = x - 1$

$\boxed{y = x - 1.}$ is normal line

Find its <u>other</u> intersection w/ $f(x)$:

$f(x) = x - 1$

$x - x^2 = x - 1$

$-x^2 + 1 = 0$

$x^2 - 1 = 0$

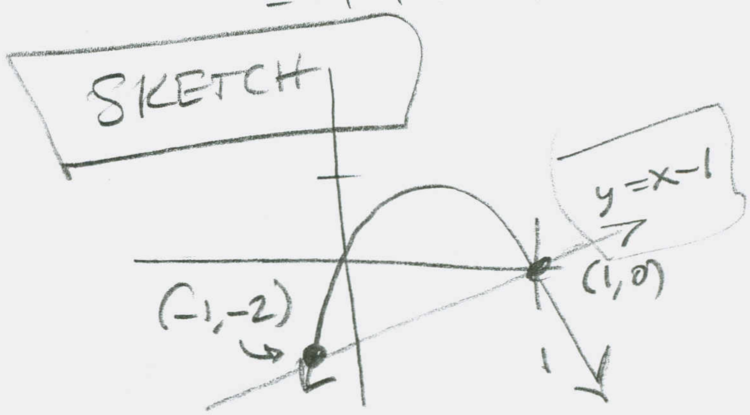
$(x-1)(x+1) = 0$

$x \in \{-1, 1\}$

New Known

$f(-1) = -1 - (-1)^2$

$= -1 - 1 = -2$



201 §3.3 II #s 84, 85, 89, 90, 91, 96, 97, 99, 100

84 $y'' + y' - 2y = x^2$ is a differential equation.

Find $A, B, C \ni Ax^2 + Bx + C$ satisfies this equation.

$y' = 2Ax + B$

$y'' = 2A$

WANT $2A + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$

$\Rightarrow 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$

$\Rightarrow 2A + B - 2C = 0$ Constant is zero

& $2A - 2B = 0$ No x-terms

& $2A = 1$ $2Ax^2 = x^2$

$\therefore \boxed{A = \frac{1}{2}} \rightarrow 2(\frac{1}{2}) - 2B = 0 \rightarrow$

$1 - 2B = 0 \rightarrow$

$1 = 2B \rightarrow$

$\boxed{\frac{1}{2} = B} \rightarrow$

$2(\frac{1}{2}) + \frac{1}{2} - 2C = 0$

$1 + \frac{1}{2} - 2C = 0$

$\frac{3}{2} = 2C$

$\boxed{\frac{3}{4} = C}$

201 §3.3 IV #s 85, 89, 90, 91, 96, 97, 99, 100

85 Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents @ $(-2, 6)$ & $(2, 0)$.

$$\boxed{f(x) = \frac{3}{16}x^3 - \frac{9}{4}x + 3}$$

$f'(x) = 3ax^2 + 2bx + c$

$f'(-2) = 3a(-2)^2 + 2b(-2) + c$

$= \underline{12a - 4b + c = 0}$

$12a - 4(0) + c = 0$

$12a + c = 0$

$c = -12a$

$f'(2) = 3a(2)^2 + 2b(2) + c$

$= \underline{12a + 4b + c = 0}$

$f(-2) = a(-2)^3 + b(-2)^2 + c(-2) + d$

$= \underline{-8a + 4b - 2c + d = 6}$

$-8a + 4(0) - 2c + 3 = 6$

$-8a - 2c = 3$

$f(2) = a(2)^3 + b(2)^2 + c(2) + d$

$= \underline{8a + 4b + 2c + d = 0}$

$8a + 4(8) + 2c - 29 = 0$

$8a + 2c = -3$

$$\begin{array}{cccc} a & b & c & d \end{array}$$

$$\left[\begin{array}{cccc|c} 8 & 4 & 2 & 1 & 0 \\ -8 & 4 & -2 & 1 & 6 \\ 12 & 4 & 1 & 0 & 0 \\ 12 & -4 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 8 & 4 & 2 & 1 & 0 \\ 0 & 8 & 0 & 2 & 6 \\ 12 & 4 & 1 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{b = 0}$

$2d = 6$

$\boxed{d = 3}$

$8a + 2(-12a) = -3$

$-16a = -3$

$\boxed{a = \frac{3}{16}}$

$c = -12a = -\cancel{12}^{3}\left(\frac{3}{\cancel{16}_{4}}\right) = \boxed{-\frac{9}{4} = c}$

201 § 3.3 II #s 89, 90, 91, 96, 97, 99, 100

89 Let $f(x) = \begin{cases} 2-x & \text{if } x \le 1 \\ x^2-2x+2 & \text{if } x > 1 \end{cases}$

Is f difbl @ $x=1$? Sketch f & f'

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -1 = f'_-(1)$$

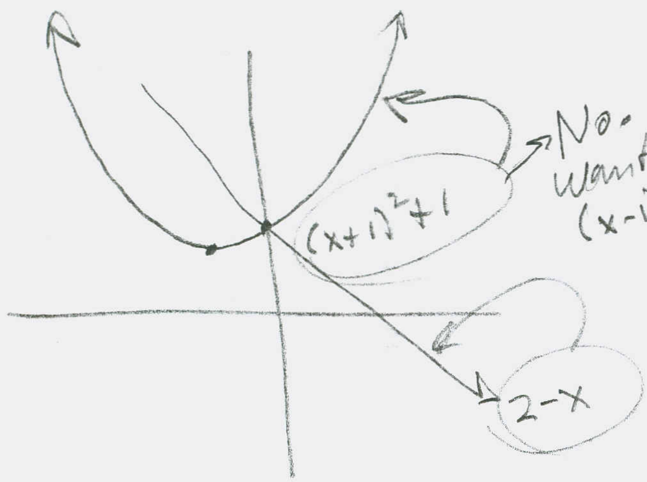
$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = (2x-2)\Big|_{x=1} = 2(1)-2 = 0 = f'_+(1)$$

$\therefore f'(1) \nexists$

$x^2-2x+2 =$

$x^2-2x+1+1 =$

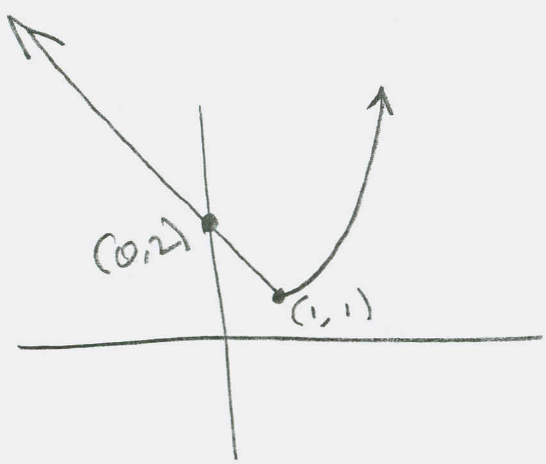
$(x-1)^2+1$



$f(1) = 2-1 = 1$

$\lim_{x\to 1^+} f(x) = 1^2-2(1)+2$

$= 1-2+2 = 1$



201 §3.3 III #s 90, 91, 96, 97, 99, 100

(90) At what numbers is the following function difbl? Give a formula for g' & sketch g & g'.

$$g(x) = \begin{cases} -1-2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$$

Check sutures:

$$\left.\begin{aligned} g'_-(-1) &= -2 \\ g'_+(-1) &= 2x\Big|_{x=-1} = -2 \end{aligned}\right\} \text{Yes } g'(-1) = -2$$

$$\left.\begin{aligned} g'_-(1) &= 2x\Big|_{x=1} = 2 \\ g'_+(1) &= 1 \end{aligned}\right\} \text{No } g'(1) \nexists$$

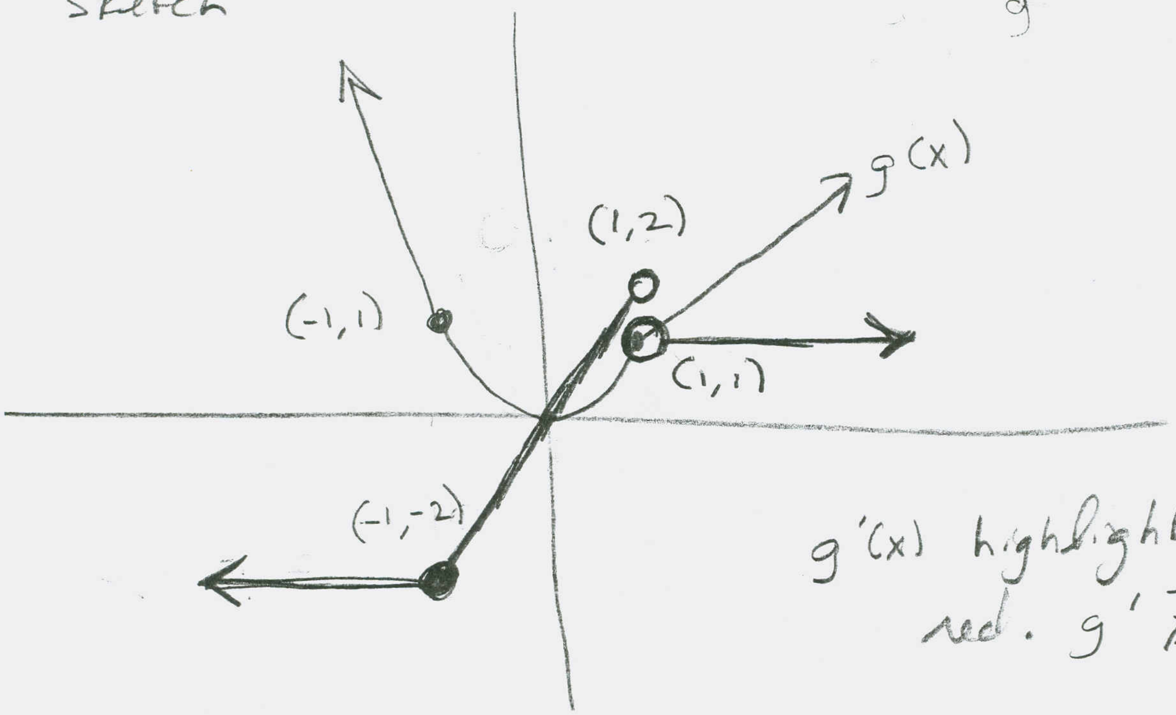
$$\boxed{g \text{ is difb}^{\underline{l}} \text{ on } \mathbb{R} \setminus \{1\}}$$

$$g'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Note: $x=1$ is left out!

201 §3.3 # 5 90, 96, 97, 99, 100

Sketch



$g'(x)$ highlighted in red. $g' \not\exists$ @ $x=1$.

$\lim_{x\to -1^-} g(x) = -1-2(-1) = 1 \rightsquigarrow (-1,1)$ hole

(96) Let $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx+b & \text{if } x > 2 \end{cases}$

Find $m, b \ni f$ is difb$^{\underline{l}}$ $\forall\, x \in \mathbb{R}$.

$x=2$ is @ issue, here

Want $f_-(2) = 2^2 = 4$ to equal

$f_+(2) = m(2)+b = 2m+b$.

This gives $\boxed{2m+b=4}$

Also want $f_-'(2) = 2x\Big|_{x=2} = 4$ to equal

$f_+'(2) = m$; This gives $\boxed{m=4}$, so $2m+b=4 \Rightarrow$

$8+b=4 \Rightarrow \boxed{b=-4}$

201 §3.3 IV #s 97, 99, 100

97) An easy proof of the Quotient Rule can be made using the PRODUCT RULE, if we assume that $F' = \left(\frac{f}{g}\right)'$ exists.

Write $f = Fg$ & use Product Rule. Then solve for F'.

<u>Proof</u> Assume, as above. Then

$$f' = F'g + Fg' \implies$$

$$F'g = f' - Fg' \implies$$

$$F' = \frac{f'}{g} - \frac{Fg'}{g} = \frac{f'}{g} - \frac{\frac{f}{g}g'}{g}$$

$$= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'}{g}\cdot\frac{g}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2} \blacksquare$$

Cute.

201 §3.3 II #s 99, 100

(99) Evaluate: $\lim\limits_{x\to 1} \dfrac{x^{1000}-1}{x-1}$.

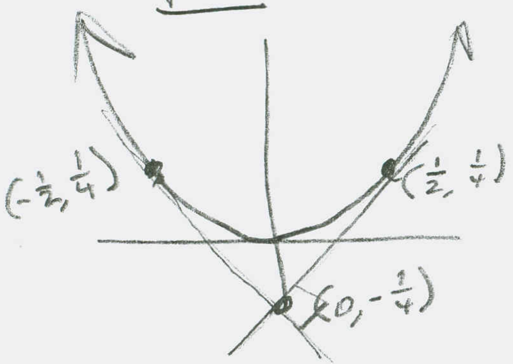
Easy. This is the derivative of $f(x)=x^{1000}$ evaluated at $x=1$

$$f'(x) = 1000x^{999} \longrightarrow$$

$$\boxed{f'(1) = 1000}$$

(100) Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to $f(x)=x^2$. Where do they intersect?

pic



I'm guessing one has slope $m=1$ & the other $m=-1$.

$f'(x)=2x \overset{\text{SET}}{=} 1 \longrightarrow \boxed{x=\frac{1}{2}}$

So $x=\frac{1}{2}$ & $x=-\frac{1}{2}$ are the sweet spot.

$f(\frac{1}{2})=(\frac{1}{2})^2=\frac{1}{4}$

$y=m(x-1)+1$

$y=1(x-\frac{1}{2})+\frac{1}{4} = x-\frac{1}{4}$

They intersect @ $(0,-\frac{1}{4})$