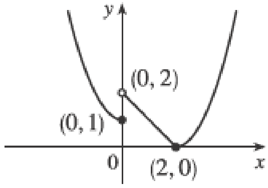
37. $f(x) = \begin{cases} 1+x^2 & \text{if } x \le 0 \\ 2-x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$



f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(1+x^2) = 1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(2-x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2-x) = 0$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

41. $f(x) = \begin{cases} cx^2+2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(cx^2+2x\right) = 4c+4$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x^3-cx\right) = 8-2c$. So f is continuous $\Leftrightarrow$ $4c+4 = 8-2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

42. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 < x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

At $x = 2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\dfrac{x^2-4}{x-2} = \lim_{x\to 2^-}\dfrac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^-}(x+2) = 2+2 = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - b + 3 = 4$, or $\mathbf{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(ax^2 - bx + 3) = 9a - 3b + 3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\mathbf{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rl} -8a + 4b = & -2 \\ 10a - 4b = & 3 \\ \hline 2a \quad\quad = & 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

46. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

49. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

60. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0. To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze Theorem $\lim_{x \to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x \to a} g(x) \neq g(a)$.

61. If there is such a number, it satisfies the equation $x^3 + 1 = x \quad \Leftrightarrow \quad x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.