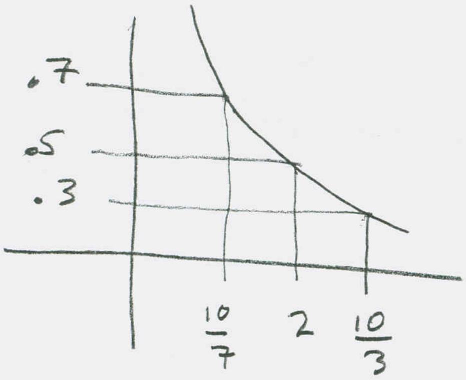
201 §2.4 #s 1, 11, 15, 16, 29, 30, 41, 42B

(1) Use the graph of $f(x) = \frac{1}{x}$ to find a number δ such that

$$0 < |x-2| < \delta \implies \left|\frac{1}{x} - .5\right| < 0.2$$



Looks like we have to stay between $\frac{10}{7}$ & $\frac{10}{3}$. The one that's closer to 2 will give us our δ:

$2 - \frac{10}{7} = \frac{14-10}{7} = \frac{4}{7}$ is smaller.

$\frac{10}{3} - 2 = \frac{10-6}{3} = \frac{4}{3}$

$$\boxed{\delta = \frac{4}{7}}$$

So if $0 < |x-2| < \frac{4}{7}$, we'll have

$$\left|\frac{1}{x} - .5\right| < .2$$

(11) A machinist is required to manufacture a disk with area 1000 cm^2.

(a) What radius produces such a disk?

$\pi r^2 = 1000 \implies$

$r^2 = \frac{1000}{\pi} \implies$

$r = \pm\sqrt{\frac{1000}{\pi}}$

So $r = \sqrt{\frac{1000}{\pi}}$ does it.

201 §2.4 #s 11, 15, 16, 29, 30, 41, 42 B

(11) b) If the error in area allowed is $\pm 5\,cm^2$, how close to the ideal radius in part (a) must we be?

Want $\left|\pi r^2 - 1000\right| < 5$

whenever $\left|r - \sqrt{\frac{1000}{\pi}}\right| < \delta$. Just

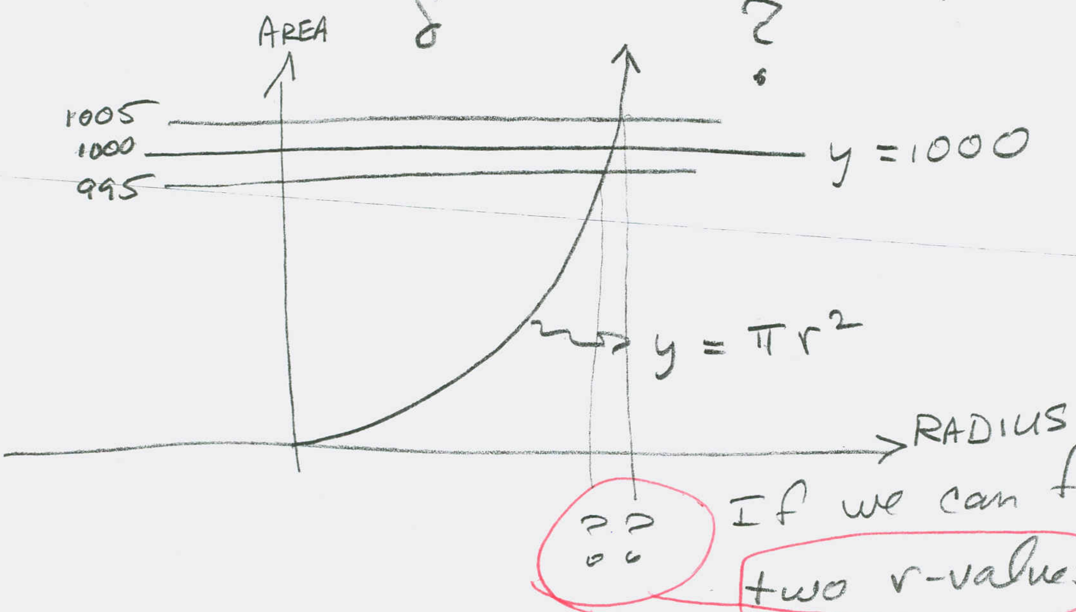
need to find δ:

Scratch:

$$\left|\pi r^2 - 1000\right| = \pi\left|\left(r^2 - \frac{1000}{\pi}\right)\right|$$

$$= \pi\left|\underbrace{\left(r - \sqrt{\frac{1000}{\pi}}\right)}_{\delta}\underbrace{\left(r + \sqrt{\frac{1000}{\pi}}\right)}_{?}\right| < 5$$

we could do a more formal thing, here, but this is getting ugly. Let's take a more graphical approach.



If we can find the two r-values where $\pi r^2 = 995$ & $\pi r^2 = 1005$, we'd be set!

201 §201 #s 11, 15, 16, 29, 30, 41, 42B

(11) $\pi r^2 = 995 \longrightarrow$

$r^2 = \frac{995}{\pi} \longrightarrow$

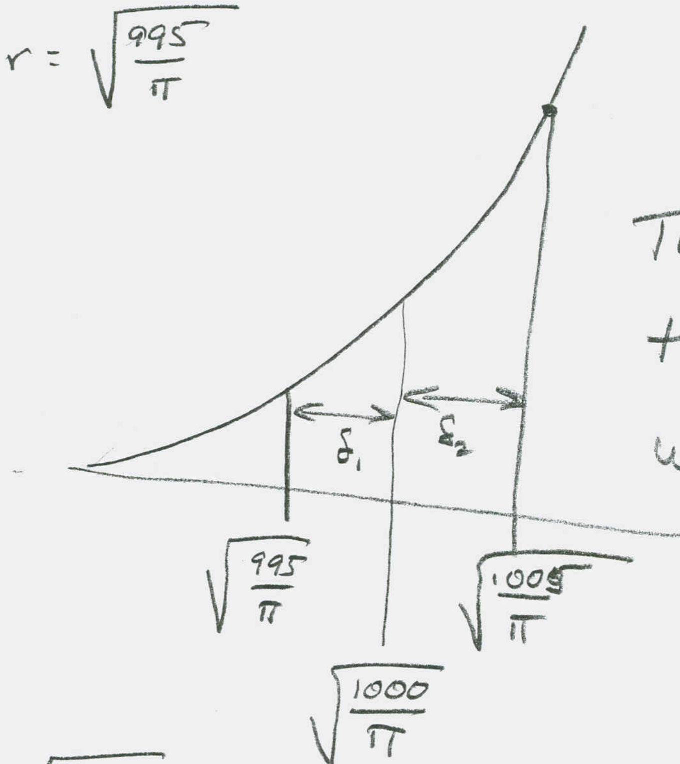
$r = \pm\sqrt{\frac{995}{\pi}}$

So, $r = \sqrt{\frac{995}{\pi}}$

$\pi r^2 = 1005 \longrightarrow$

$\vdots$

$r = \sqrt{\frac{1005}{\pi}}$



The smaller of the two δ's will keep us within $5\,cm^2$ of Area = $1000\,cm^2$

δ_1: $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}}$

$= \frac{\sqrt{1000}}{\sqrt{\pi}} - \frac{\sqrt{995}}{\sqrt{\pi}} = \frac{\sqrt{1000} - \sqrt{995}}{\sqrt{\pi}} \approx .0446589966$

δ_2: $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx .04454 7488$

$\boxed{\text{He must make his radius to within } \pm .04454 7488 \text{ of ideal radius in order to keep area within } 5\,cm^2 \text{ of } 1000\,cm^2}$

201 § 2.4 #s 11, 15, 16, 29, 30, 41, $\boxed{42B}$

(11) (c) In terms of ε-δ defin,

$\lim_{x \to a} f(x) = L$,

x is the radius r

$f(x)$ is the area πr^2

a is the ideal radius $r = \sqrt{\frac{1000}{\pi}}$

L is the ideal area $L = 1000 \text{ cm}^2$

ε is 5 cm^2, and

δ is the $.04454748 \text{ cm}$.

To be safe, we'd probably go smaller, like $\delta = .04$.

#s 15, 16 we prove the statement using the ε-δ definition of limit. Draw the picture.

(15) $\lim_{x \to 1} (2x+3) = 5$

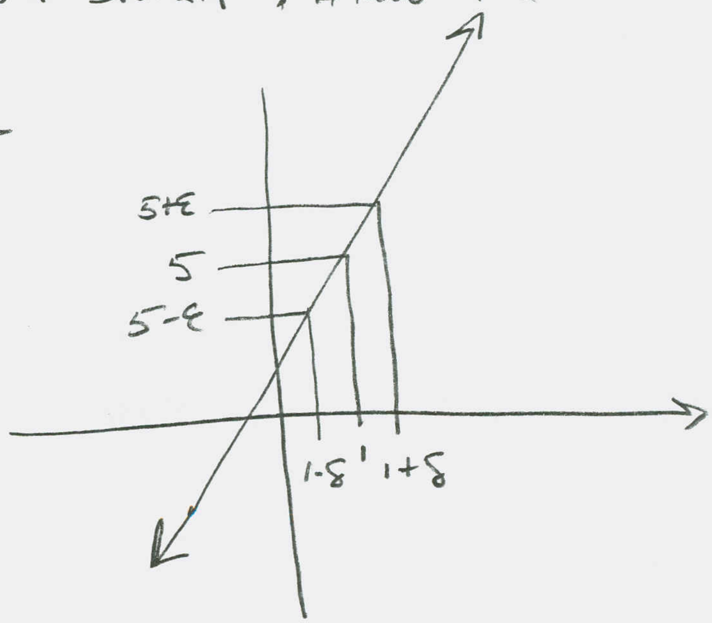
Scratch:

want $|2x+3-5| < \varepsilon$

$|2x-2| < \varepsilon$

$2|x-1| < \varepsilon$

$|x-1| < \frac{\varepsilon}{2} \equiv \delta$



201 § 2.4 #s 15, 16, 29, 30, 41, 42B

(15) <u>Proof</u> Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{2}$. Then if $0 < |x-1| < \delta$, we have

$$|2x+3-5| = |2x-2| = 2|x-1| < 2\delta = \varepsilon. \blacksquare$$

(16) <u>Claim:</u> $\lim_{x \to -2} \left(\frac{1}{2}x + 3\right) = 2$

<u>Scratch:</u>

want $\left|\frac{1}{2}x+3-2\right| < \varepsilon$

$$\left|\frac{1}{2}x+1\right| < \varepsilon$$

$$\frac{1}{2}|x+2| < \varepsilon$$

$$|x+2| < 2\varepsilon$$

$$|x-(-2)| < 2\varepsilon \equiv \delta$$

<u>Proof</u> Let $\varepsilon > 0$ be given. Define $\delta = 2\varepsilon$. Then if $0 < |x-(-2)| < \delta$, we have

$$\left|\frac{1}{2}x+3-2\right| =$$

$$\left|\frac{1}{2}x+1\right| =$$

$$\frac{1}{2}|x+2| =$$

$$\frac{1}{2}|x-(-2)| < \frac{1}{2}\delta$$

$$= \frac{1}{2}\cdot\frac{\varepsilon}{2} = \varepsilon \blacksquare$$

201 § 2.4 #s 29, 30, 41, $\boxed{42B}$

#s 19-32 Prove, using ε-δ defn of limit.

29 $\lim_{x \to 2} (x^2 - 4x + 5) = 1$

<u>Scratch</u> want $|x^2 - 4x + 5 - 1| < \varepsilon$

$|x^2 - 4x + 4| < \varepsilon$

$\underbrace{|x-2|}_{\delta}^2 < \varepsilon$

$|x-2| < \sqrt{\varepsilon} \equiv \delta$

<u>Proof</u>

Let $\varepsilon > 0$ be given. Define $\delta = \sqrt{\varepsilon}$.

Then $0 < |x-2| < \delta \implies$

$|x^2 - 4x + 5 - 1| = |x^2 - 4x + 4| = |x-2|^2 < \delta^2 = \varepsilon$ ∎

201 §2.4 #s 30, 41, $\boxed{42B}$

30 More standard one than #29

Claim:

$$\lim_{x\to 3}(x^2+x-4)=8$$

Scratch

Want $|x^2+x-4-8|<\varepsilon$

$|x^2+x-12|<\varepsilon$

$|x+4|\underbrace{|x-3|}_{\delta}<\varepsilon$

We need an upper bound on $|x+4|$. The lever we use is to ASSUME $\delta\le 1$.

Since we're looking at an interval centered @ $a=3$, this means

$|x-3|<1$

$$\begin{array}{ccccc} -1 & < & x-3 & < & 1 \\ +7 & = & +7 & = & +7 \\ \hline 6 & < & x+4 & < & 8, \text{so} \end{array}$$

Now make an "x+4" out of the "x-3" by addition

that $|x+4|<8$. Use the 8:

201 §2.4 #s 30, 41, 42 B

30 Scratch, cnt'd

$|x+4||x-3| < \varepsilon$

$8|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{8}$

The proof has one small wrinkle:

PROOF Let $\varepsilon > 0$ be given. Define

$\delta = \min\{1, \frac{\varepsilon}{8}\}$. Then if $0 < |x-3| < \delta$,

we have

$|x^2+x-4-8| = |x^2+x-12| = |x+4||x-3|$

$< 8\delta \leq \varepsilon$ ■

41 How close to -3 do we have to take x

so that $\frac{1}{(x+3)^2} > 10000$?

$1 > 10000(x+3)^2$

$\frac{1}{10000} > (x+3)^2$

$\underbrace{|x+3|}_{\text{Distance from } x \text{ to } -3} < \boxed{\sqrt{\frac{1}{10000}}}$ closer than

201 §2.4 #42B

42 Prove formally that $\lim_{x \to -3} \frac{1}{(x+3)^2} = \infty$

<u>Scratch</u> Let M be given.

Want $\frac{1}{(x+3)^2} > M$ by $x \longrightarrow -3$

$$\frac{1}{(x+3)^2} > M$$

$$\frac{1}{M} > (x+3)^2$$

$$(x+3)^2 < \frac{1}{M}$$

$$\underbrace{|x+3|}_{\text{Distance to } -3.} < \underbrace{\sqrt{\frac{1}{M}}}_{\delta}$$

<u>Proof</u>

Let $M > 0$ be given.

Define $\delta = \sqrt{\frac{1}{M}}$. Then

$0 < |x+3| < \delta \Longrightarrow$

$$\frac{1}{(x+3)^2} > \frac{1}{\delta^2} = \frac{1}{\left(\sqrt{\frac{1}{M}}\right)^2}$$

$$= \frac{1}{\left(\frac{1}{M}\right)} = M \quad \blacksquare$$