$f(x) = b^x$ where $b > 1$ is exponential growth.

$f'(x)$:

$$\frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = b^x\left(\frac{b^h - 1}{h}\right) = b^x\left(\frac{b^h - b^0}{h}\right)$$

$$= \frac{b^h - b^0}{h} b^x \xrightarrow{h \to 0} f'(0) b^x$$

If only we knew what $f'(0)$ was!

It'd be REALLY nice if $f'(0) = 1$

$2 < b < 3$

The ideal # is the "Natural Exponential" grows as fast as it is tall.

$f'(0) = 1$

$e$ stands for "Euler."

$f(x) = e^x$ $\qquad e \approx$ 2.718281828

$e$ is SUPER-DUPER FANTASTIC!

$$\frac{d}{dx}[e^x] = e^x$$

$\frac{d}{dx}[b^x] = ?$ We'll need logarithms for that

$g(x) = b^x \Rightarrow g'(x) = g'(0) b^x$

Differential Equations

$y' = y$ $\qquad \frac{dy}{dx} - y = 0$

or $y' - y = 0$

$\Rightarrow y = ce^x$

$y' = ce^x$

---

Chain Rule Version:

$$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)} u'(x) = u'(x) e^{u(x)} = u' e^u$$

$$\frac{d}{dx}\left[e^{\sin(x)}\right] = \cos(x) e^{\sin(x)}$$

§6.3 Logarithmic Functions

Define $\log_b(x)$ to be the Inverse of $f(x)=b^x$

Book: $y=\log_b(x)$ means $x=b^y$

ME: $\log_b(b^x) = x$

$b^{\log_b(x)} = x$

$\log_e(x) = \ln(x)$ = natural log function.

$\ln(e^x) = x$ , $e^{\ln(x)} = x$

$7 = e^{\ln(7)}$

$7^x = \left(e^{\ln(7)}\right)^x = e^{(\ln(7))x}$

$\frac{d}{dx}[5x] = 5$

$\frac{d}{dx}[(\ln(7))x] = \ln(7)$

$\frac{d}{dx}[7^x] = \frac{d}{dx}\left[e^{(\ln(7))x}\right] = \ln(7)e^{(\ln(7))x}$

$= (\ln(7))\left(e^{\ln(7)}\right)^x = (\ln(7))7^x$

$\frac{d}{dx}[b^x] = (\ln(b))b^x$

$\frac{d}{dx}[5^x] = (\ln(5))5^x$

$$\boxed{y=\ln(x)} \Rightarrow$$

$$e^y = x$$

$$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$$

$$e^y \cdot y' = 1 \Rightarrow$$

$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \boxed{\frac{1}{x} = y'}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

$$\int \frac{dx}{x} = \ln(x) + C$$

for $x > 0$.

In general,

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\frac{d}{dx}\left[\ln(\sin(x))\right] = \frac{1}{\sin(x)} \cdot \cos(x) = \cot(x)$$

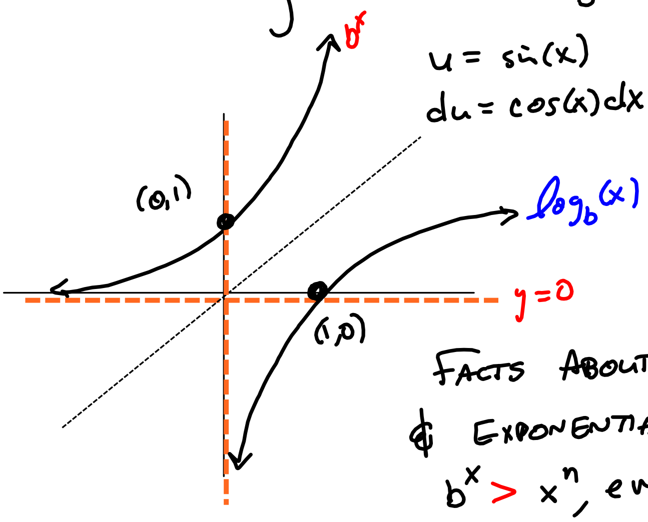
Think Substitution Rule:

$$\int \cot(x)\,dx = \int \frac{\cos(x)\,dx}{\sin(x)} = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sin(x)| + C$$

$u = \sin(x)$

$du = \cos(x)\,dx$




$b^x$ $(b>1)$

$\mathcal{D} = \mathbb{R}$

$\mathcal{R} = (0,\infty)$

H.A. $y=0$

Increasing

$\log_b(x)$

$\mathcal{D} = (0,\infty)$

$\mathcal{R} = \mathbb{R}$

V.A. $x=0$

Increasing

FACTS ABOUT LOGARITHMIC GROWTH & EXPONENTIAL GROWTH.

$b^x > x^n$, eventually.

$\log_b(x) < x^n$, . . .

$$\lim_{x\to\infty} \frac{x^5}{1.2^x} = 0, \text{ etc.}$$

**2 Theorem** If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $b^x > 0$ for all $x$. If $0 < b < 1$, $f(x) = b^x$ is a decreasing function; if $b > 1$, $f$ is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

**1.** $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

① $\log_b(xy) = \log_b(x) + \log_b(y)$

② $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

③ $\log_b(r^x) = x\log_b(r)$

BIG SKILL

Logarithmic Differentiation

$y = \sin(x)^{\cos(x)} \Rightarrow$

$\ln(y) = \ln\left(\sin(x)^{\cos(x)}\right) = \cos(x)\ln(\sin(x))$

$\frac{1}{y} \cdot y' = -\sin(x)\ln(\sin(x)) + \cos(x)\left(\frac{1}{\sin(x)}\right)\cos(x)$

$\Rightarrow y' = \left(-\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right)\underbrace{\left(\sin(x)^{\cos(x)}\right)}_{y}$

$$y = \ln\left(\frac{\sqrt{2x^3 - \sin(x)}}{\cot(x)(x^4 - 3x^3 + 7)}\right)$$

what's $y'$?

$$y = \underbrace{\ln\left(\sqrt{2x^3 - \sin(x)}\right)}_{\frac{1}{2}\ln(2x^3 - \sin(x))} - \ln(\cot(x)) - \ln(x^4 - 3x^3 + 7)$$

$$\Rightarrow y' = \frac{1}{2}\left(\frac{6x^2 - \cos(x)}{2x^3 - \sin(x)}\right) - \left(\frac{-\csc^2(x)}{\cot(x)}\right) - \left(\frac{4x^3 - 9x^2}{x^4 - 3x^3 + 7}\right)$$

POIFECK!

Find $f^{-1}$

$f(x)=\ y=(\ln(x))^3 \longrightarrow$

$(\ln(y))^3=x$

$e^{\ln(y)}=e^{\sqrt[3]{x}}$

$y=e^{\sqrt[3]{x}}=f^{-1}(x)$

$y=\frac{3\ln(x)}{x}$ Find tangent line @ $(1,0)$, $(e,\frac{3}{e})$

$y'=\frac{f'g-fg'}{g^2}=\frac{3(\frac{1}{x})\cdot x-3\ln(x)(1)}{x^2}$

$y'(1)=\frac{3-0}{3}=1 \rightarrow 3$, $3 \rightarrow 1^2$

$y=1(x-1)+0 \quad =\boxed{3(x-1)=y}$, $1 \rightarrow 3$

$y'(e)=\frac{3-3}{\text{~~}}=0$

$\boxed{\cancel{y=0}}$ No. y isn't zero It has zero slope & passes thru. $(e,\frac{3}{e})$

So, $y=0(x-e)+\frac{3}{e}=\boxed{\frac{3}{e}=y}$

$y=\frac{3\ln(x)}{x}=(3\ln(x))(\frac{1}{x})=(3\ln(x))\,x^{-1}$

$y'=f'g+fg'=\frac{3}{x}\cdot x^{-1}+(3\ln(x))(-1x^{-2})$

$=\frac{3-3\ln(x)}{x^2}$