Remind me to hit record!

Today: Finish graphing

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined. BAD: $\sqrt[2n]{\text{neg}}$, $\frac{\text{stuff}}{0}$

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 3.4 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 1.5 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

1
$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$
$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

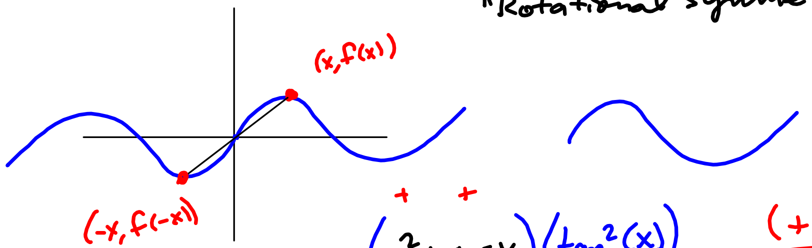
G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

ODD Func: $f(-x) = -f(x)$

Symmetry thru the origin

"Rotational Symmetry"



$$\frac{\overset{+}{(x^2+\cos x)}\overset{+}{(\tan^2(x))}}{\underbrace{\underset{-}{(\sin(x))}\underset{-}{(x-\tan(x))}}_{-}} = \frac{(+)(+)}{(-)(-)} = + \text{ Even}$$

Recall, the definition on Page 18/19 of "increasing and decreasing on an interval."

f increasing (decreasing) on an interval I

if $\forall\ x_1 < x_2 \in I$, we have $f(x_1) < f(x_2)$ · $(f(x_1) > f(x_2))$

This allows overlap between increasing & decreasing.

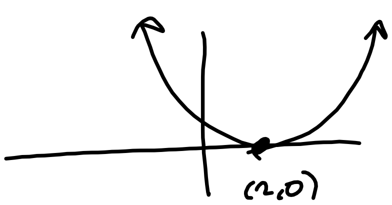
$y=(x-2)^2$ decreasing on $(-\infty,2]$ } overlap @ $x=2$.

increasing on $[2,\infty)$

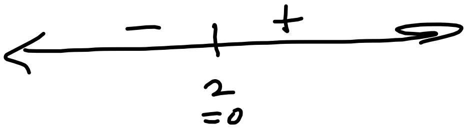
C3, $f'(x) > 0 \Rightarrow$ increasing

$f'(x) < 0 \Rightarrow$ decreasing

So, C3 thinks $f(x) = (x-2)^2$ is decreasing on $(-\infty,2)$ & increasing on $(2,\infty)$



$f'(x) = 2(x-2) = 2x-4 = 0 \Rightarrow x=2$

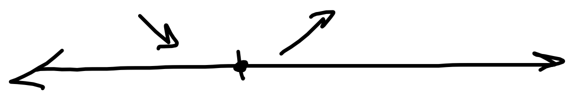


§3.3

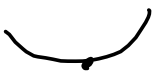
We soft peddled

1^{st} deriv. test. $f(7)\ \exists$, $f'(x) < 0$ for $x < 7$, $f'(x) > 0$ for $x > 7$

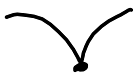
$(7, f(7))$ is a local min on the graph



7



$(7, f(7))$



$(7, f(7))$

$(x-7)^{2/3}$

$f'(7)\ \nexists$ in the case of a cusp.

Similarly for local max

max

2^{nd} Derivative test $f'(7) = 0$

Needs to be smooth

$f''(7) > 0$  Min

$f''(7) < 0$  MAX

Requires $f', f''\ \exists$.

A graphing calculator is recommended. → Yes, indeed!

For the limit

$$\lim_{x\to\infty} \frac{1-6x}{\sqrt{x^2+1}} = -6$$

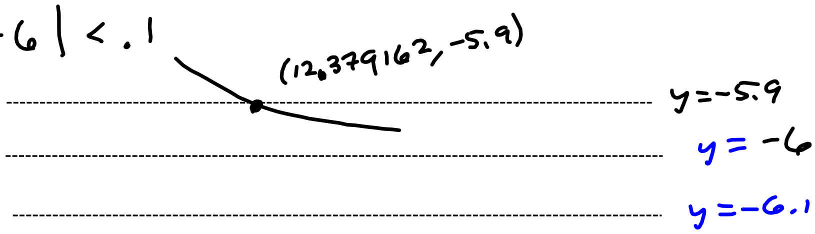
illustrate the definition by finding the smallest integer values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

$\varepsilon = 0.1$ $N =$ 13

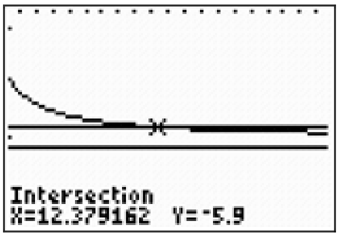
$\varepsilon = 0.05$ $N =$ 23

Want $$-6-.1 < \frac{1-6x}{\sqrt{x^2+1}} < -6+.1$$

$$\left|\frac{1-6x}{\sqrt{x^2+1}} + 6\right| < .1$$



$$\left|\frac{1-6x+6\sqrt{x^2+1}}{\sqrt{x^2+1}}\right| < .1$$



$$-.1 < \frac{1-6x+6\sqrt{x^2+1}}{\sqrt{x^2+1}} < .1$$

IOW: Graph $\frac{1-6x}{\sqrt{x^2+1}} = y$, $y=-5.9$, $y=-6.1$

$$y' = \frac{-6\sqrt{x^2+1} - (1-6x)\left(\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right)}{x^2+1}$$

$y'' =$ Owie!

Another thing to try:

$$\frac{1-6x}{\sqrt{x^2+1}} = -5.9$$

$1-6x = -5.9\sqrt{x^2+1}$ square both sides

$36x^2-12x+1 = (5.9)^2(x^2+1)$ is a messy quadratic, but do-able.

Let $a = (5.9)^2$

$36x^2-12x+1 = ax^2+a$

$(36-a)x^2-12x+(1-a)=0$

Let $d=a$

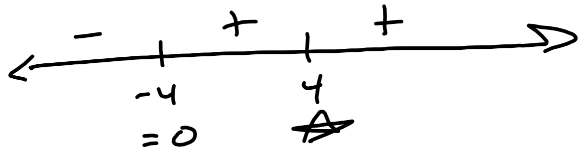
$(36-d)x^2-12x+(1-d)=0$

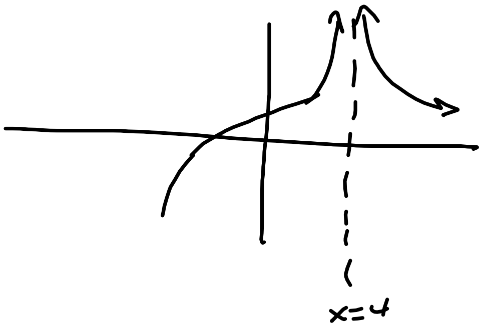
$a=36-d$, $b=-12$, $c=1-d$

$b^2-4ac =$ etc. Do-able but painful.

Use the guidelines of this section to sketch the curve.

$$y = \frac{(x+4)^3}{(x-4)^2}$$





1
1 1
1 2 1
1 3 3 1

$$\frac{(x+4)^3}{(x-4)^2} = \frac{x^3 + 3(4)x^2 + 3(4)^2x + 4^3}{x^2 - 8x + 4} = \frac{x^3 + 12x^2 + 48x + 64}{\text{same}}$$

$$\begin{array}{r l} & x + 20 \quad \text{DONE with Asymptote} \\ x^2 - 8x + 4 \; \big) & x^3 + 12x^2 + 48x + 64 \\ & -(x^3 - 8x^2 + 4x) \\ \hline & 20x^2 \end{array}$$

Slant Asymptote:

$$y = x + 20$$

$$\begin{array}{r l} & x + 20 \quad r \; 204x - 16 \\ x^2 - 8x + 4 \; \big) & x^3 + 12x^2 + 48x + 64 \\ & -(x^3 - 8x^2 + 4x) \\ \hline & 20x^2 + 44x + 64 \\ & -(20x^2 - 160x + 80) \\ \hline & 204x - 16 \end{array}$$

This says

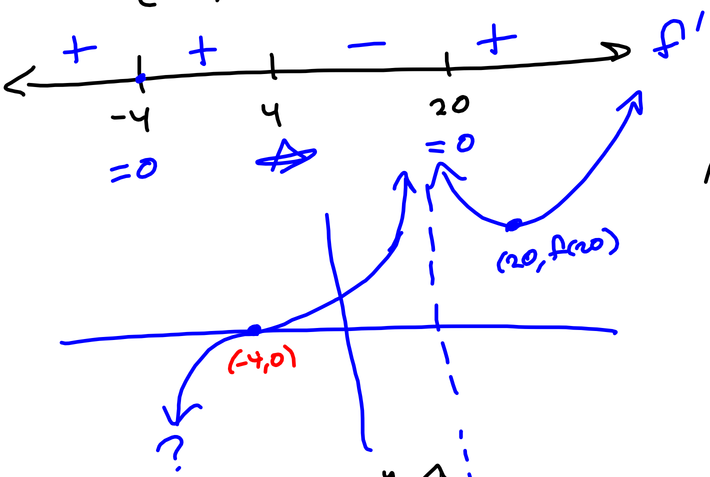
$$f(x) = x + 20 + \frac{204x - 16}{(x-4)^2}$$

→ 0 as $x \to \pm\infty$

$$f(x) = \frac{(x+4)^3}{(x-4)^2} \Rightarrow f'(x) = \frac{3(x+4)^2(x-4)^2 - (x+4)^3(2(x-4))}{(x-4)^4}$$

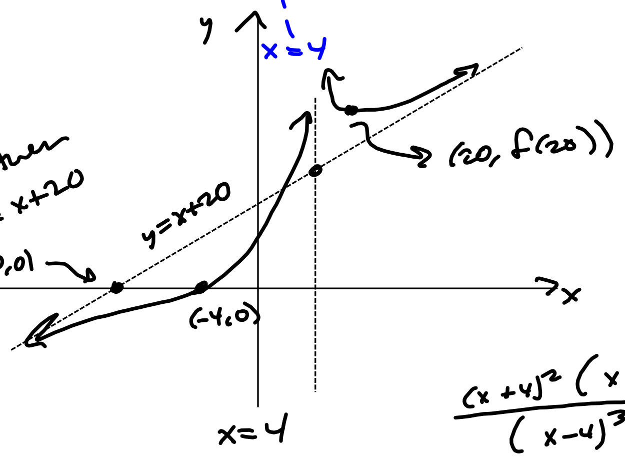
$$= \frac{(x+4)^2(x-4)\left[3(x-4) - 2(x+4)\right]}{(x-4)^4}$$

$$= \frac{(x+4)^2\left[3x-12-2x-8\right]}{(x-4)^3} = \frac{(x+4)^2(x-20)}{(x-4)^3} = f'$$



Missing my slant asymptote.

It's got some wiggle, crossing y=x+20 & then approach→ y=x+20



$$\frac{(x+4)^2(x-20)}{(x-4)^3}$$

$$f''(x) = \frac{\left(2(x+4)(x-20) + (x+4)^2\right)(x-4)^3 - (x+4)^2(x-20)\left(3(x-4)^2\right)}{(x-4)^6}$$

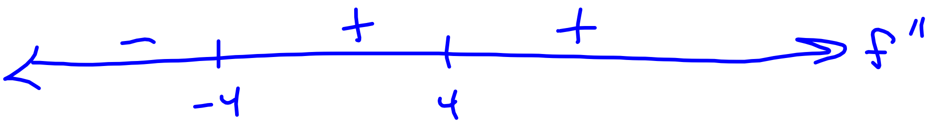
$$= \frac{(x+4)(x-4)^2\left[\left(2(x-20) + (x+4)\right)(x-4) - (x+4)(x-20)(3)\right]}{(x-4)^6 \to 4}$$

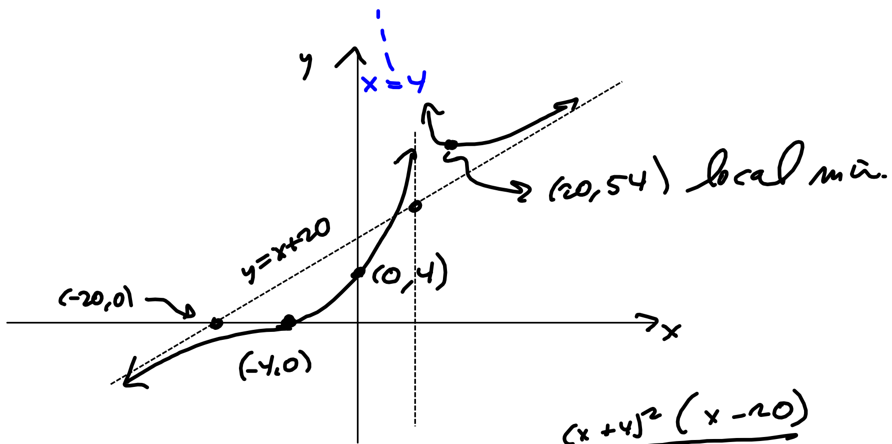
$$= \frac{(x+4)\left[(2x-40+x+4)(x-4) - 3(x^2-16x-80)\right]}{(x-4)^4}$$

$$= \frac{(x+4)\left[(3x-36)(x-4) - 3(x^2-16x-80)\right]}{(x-4)^4}$$

$$= \frac{3(x+4)\left[(x-12)(x-4) - x^2+16x+80\right]}{(x-4)^4}$$

$$= \frac{3(x+4)\left[x^2-16x+48 - x^2+16x+80\right]}{(x-4)^4} = \frac{3(x+4)[128]}{(x-4)^4}$$





$$\frac{(x+4)^2(x-20)}{(x-4)^3}$$

$x=4$

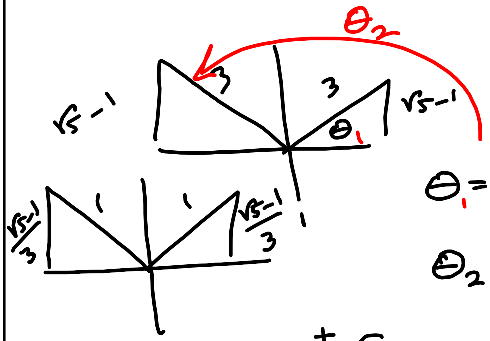
$$f(20)=\frac{(20+4)^3}{(20-4)^2}=\frac{24^3}{16^2}$$

2 | 24
2 | 12
2 | 6
3

2 | 16
2 | 8
2 | 4
2

$$\frac{(2^3\cdot 3)^3}{(2^4)^2}=\frac{2^9\cdot 3^3}{2^8}=2\cdot 3^3=54$$

$$\sin(x) = \frac{-1 \pm \sqrt{5}}{3} \to \frac{-1+\sqrt{5}}{3}, \quad \frac{-1-\sqrt{5}}{3} < -1 \; \times$$



$$\theta_1 = \arcsin\left(\frac{\sqrt{5}-1}{3}\right)$$

$$\theta_2 = \pi - \arcsin\left(\frac{\sqrt{5}-1}{3}\right)$$

$$\frac{1 \pm \sqrt{5}}{3} \to \frac{1+\sqrt{5}}{3} \; \times, \quad \frac{1-\sqrt{5}}{3}$$

$$\theta \in [0, 2\pi]$$

$$\theta_1 = \arcsin\left(\frac{1-\sqrt{5}}{3}\right) < 0$$

$$2\pi + \arcsin\left(\frac{1-\sqrt{5}}{3}\right)$$