$R(x) = \dfrac{2x^2 - x - 15}{x^2 - 4x - 21}$. we graph $R(x)$ using MAT 121 techniques.

$\mathcal{D}$: $x^2 - 4x - 21 = (x-7)(x+3) \overset{\text{SET}}{=} 0 \rightarrow x \in \{-3, 7\} \Rightarrow \mathcal{D} = \mathbb{R} \setminus \{-3, 7\}$

Vertical Asymptotes: $\boxed{x = -3,\ x = 7}$ V.A.

x-int: $2x^2 - x - 15 = (2x+5)(x-3) \overset{\text{SET}}{=} 0 \longrightarrow$

$x \in \{-\frac{5}{2}, 3\} \rightsquigarrow \boxed{(-\frac{5}{2}, 0),\ (3, 0) \text{ x-INTS}}$

y-int: $\dfrac{-15}{-21} = \dfrac{-5}{-7} = \dfrac{5}{7} \rightsquigarrow \boxed{(0, \frac{5}{7}) \text{ y-int}}$

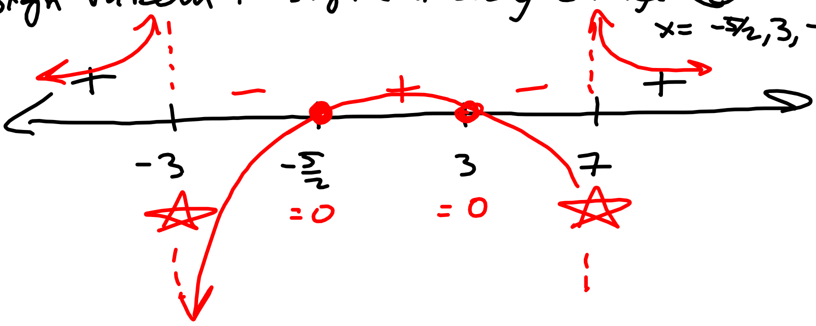
Horizontal Asymptote: $\boxed{y = 2}$ H.A. $\quad R(x) = \dfrac{(2x+5)(x-3)}{(x-7)(x+3)}$

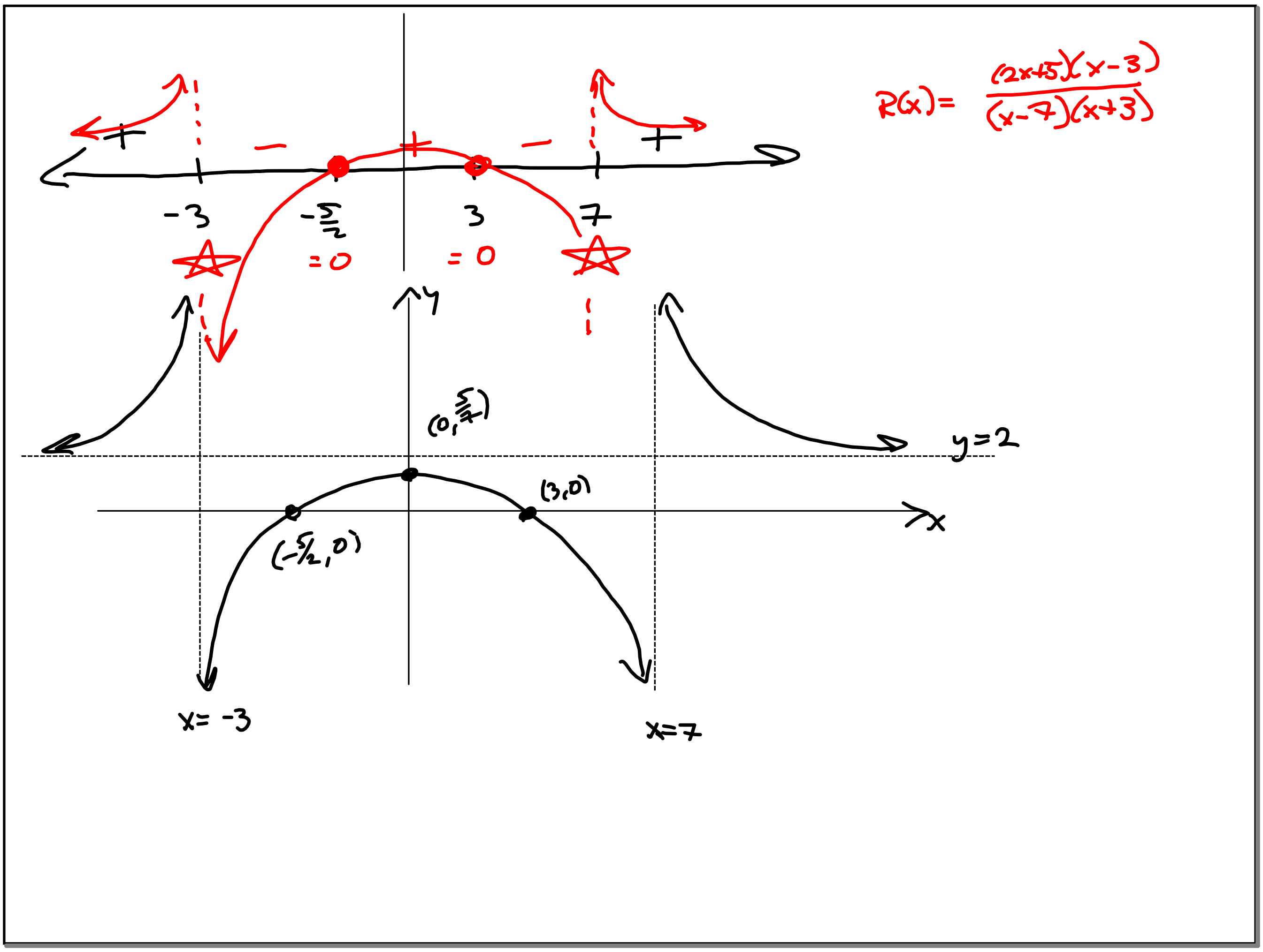
from $\dfrac{2x^2}{x^2} = 2 = y$

Sign Pattern: Sign can only change @ x-ints or V.A.

$x = -5/2, 3, -3, 7 \rightsquigarrow$

$-3, -\frac{5}{2}, 3, 7$




$R(x)=\frac{(2x+5)(x-3)}{(x-7)(x+3)}$
+
−
+
−
+
−3
$-\frac{5}{2}$
= 0
3
= 0
7
y
$(0,\frac{5}{7})$
y=2
$(-\frac{5}{2},0)$
(3,0)
x
x= −3
x=7

Bonus for College Algebra :

Find where R(x) intersects its H.A.

$$R(x) = \frac{2x^2 - x - 15}{x^2 - 4x - 21} \overset{\text{SET}}{=} 2\left(\frac{x^2 - 4x - 21}{x^2 - 4x - 21}\right) = \frac{2x^2 - 8x - 42}{x^2 - 4x - 21} \Longrightarrow$$

$$2x^2 - x - 15 = 2x^2 - 8x - 42$$

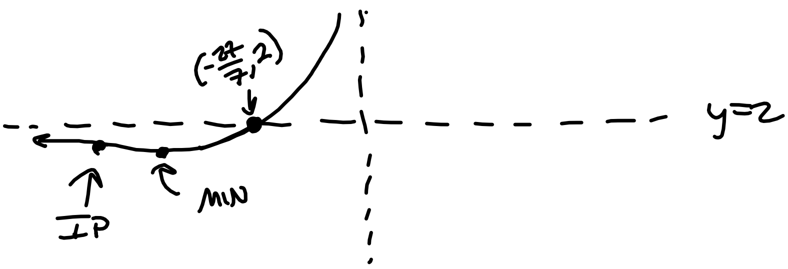
(idiot)

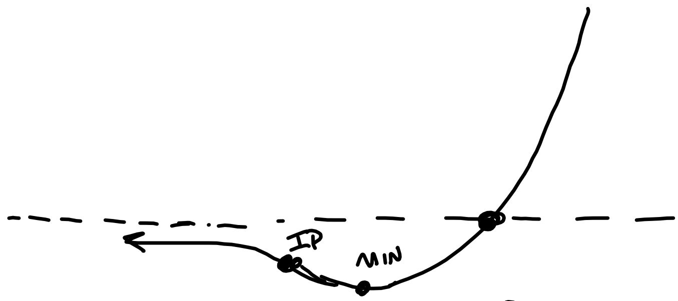
$$-x - 15 = -8x - 42 \Longrightarrow$$

$$7x = -27 \Longrightarrow$$

$x = -\frac{27}{7} \rightsquigarrow \left(-\frac{27}{7}, 2\right)$ is intersection.

So, the left part of the graph is :





Approaching y=2 from BELOW says it's concave Down, eventually.

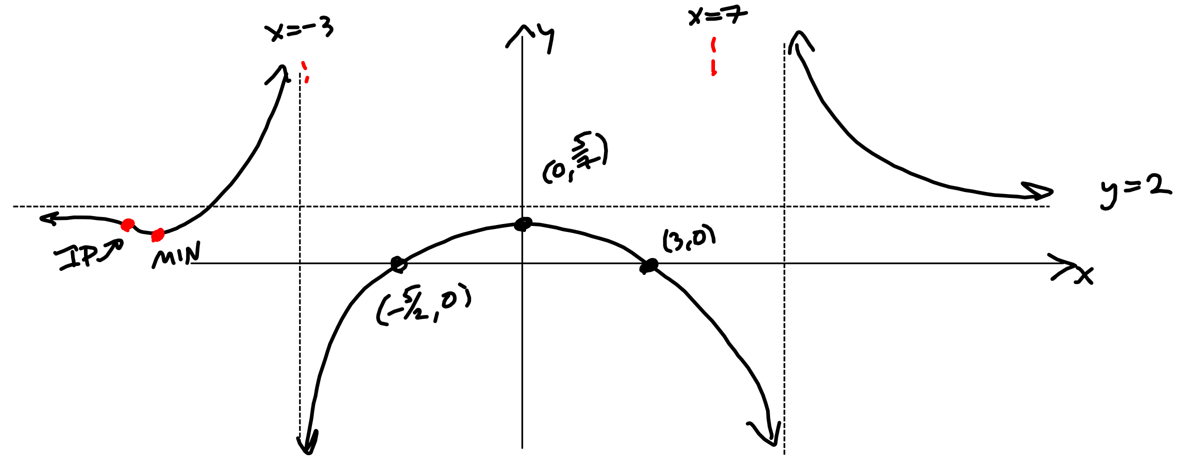
Min: (-6.907736643, 1.607083130) approx

IP: (-10.51278929, 1.645895250) (approx.)

$$\frac{14x^3 + 162x^2 + 234x + 822}{(x^2 - 4x - 21)^3} = R''(x), \text{ according to Maple.}$$

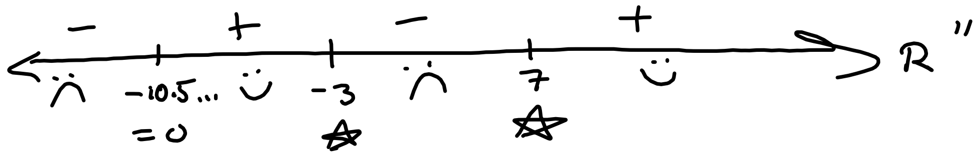
$R''(x) = 0 \Longrightarrow x \approx -10.51278929$

No shame in using technology.



Min: (-6.907736643, 1.607083130) approx

IP: (-10.51278929, 1.645895250) (approx.)



$$\frac{14x^3+162x^2+234x+822}{(x^2-4x-21)^3} = \frac{x^3+\cdots}{(x+3)(x-7)}$$

Oblique Asymptote

$$\frac{x^2-5x-14}{x-3} = R(x)$$

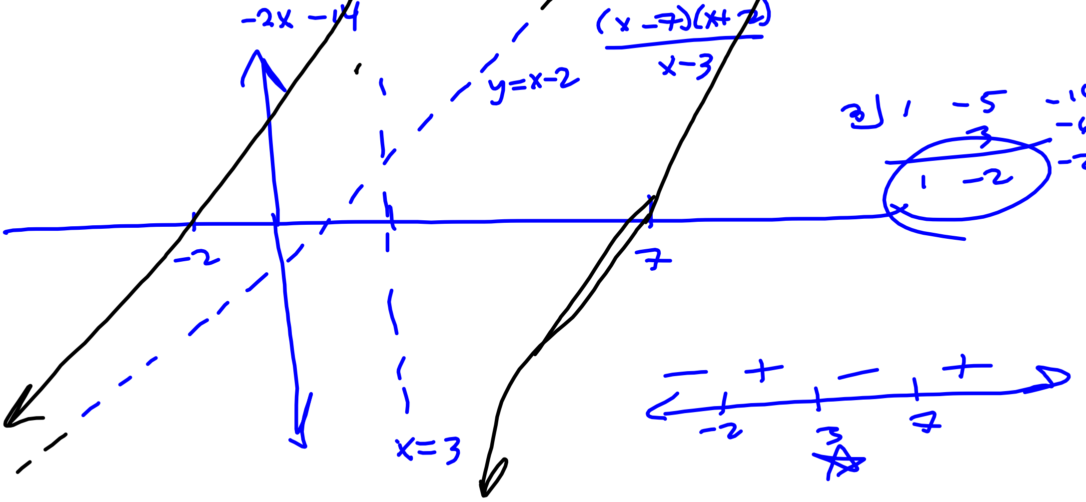
How to get it: Long Division:

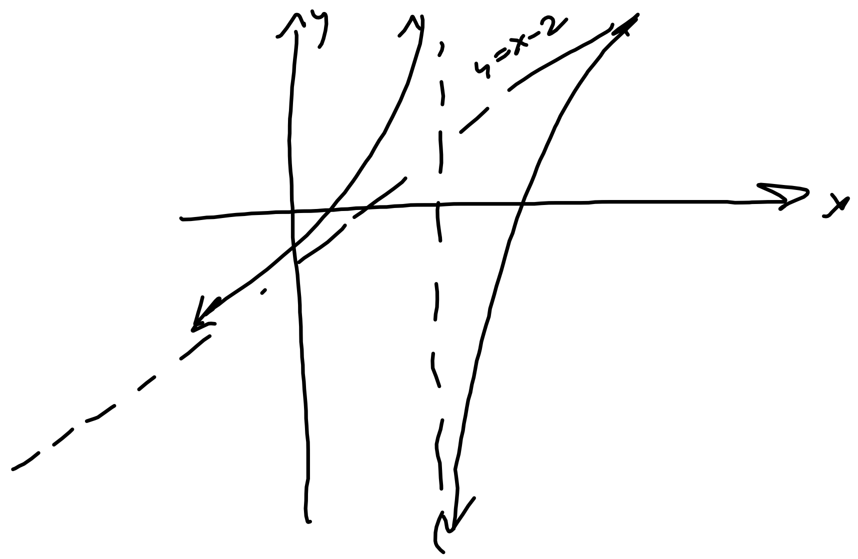
$$\begin{array}{r|l} & x-2 \\ \hline x-3 & x^2-5x-14 \\ & -(x^2-3x) \\ \hline & -2x-14 \end{array}$$

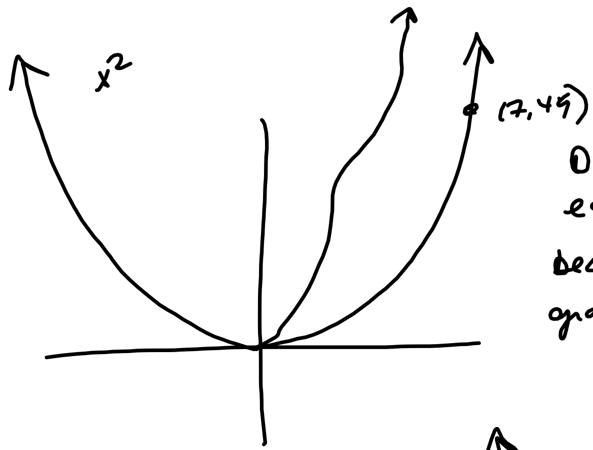
This has oblique asymptote $y = x-2$.

$$\frac{(x-7)(x+2)}{x-3}$$

$$\begin{array}{r|rrr} 3 & 1 & -5 & -14 \\ & & 3 & -6 \\ \hline & 1 & -2 & -20 \end{array}$$

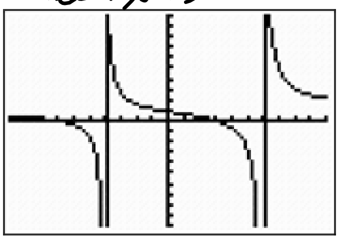


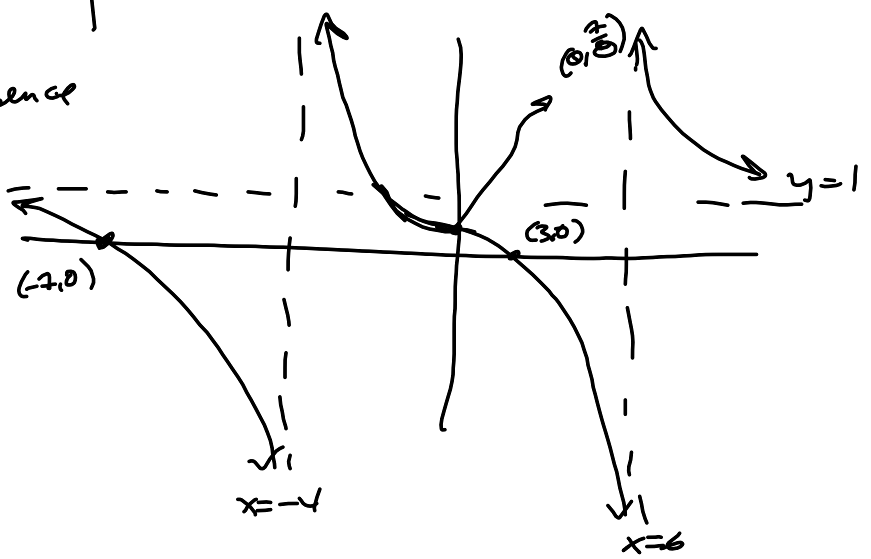




Don't Lose the essence of the shape because of your graphing tech.

Tech loses the essence





$$R(x) = \frac{x^2-5x-14}{x-3} \quad (x-7)(x+2)$$

$\mathcal{D} = \mathbb{R} \setminus \{3\} \rightarrow \boxed{\text{V.A. } x=3}$

$\boxed{\text{x-int: } (-2,0), (7,0)}$

This is a continuation of a previous example.

No H.A., but O.A.

$$\begin{array}{r|rrr} 3 & 1 & -5 & -14 \\ & & 3 & -6 \\ \hline & 1 & -2 & -20 \end{array}$$

This says: $R(x) = x-2+\frac{-20}{x-3}$

$\frac{29}{3} = 9 \text{ r } 2$ says: $29 = 3(9)+2$

$\frac{29}{3} = 9+\frac{2}{3}$

$\boxed{y = x-2 \text{ is O.A.}}$

$$\frac{f'g-fg'}{g^2} = \left(\frac{f}{g}\right)'$$

$$R'(x) = \frac{(2x-5)(x-3)-(x^2-5x-14)}{(x-3)^2} = \frac{2x^2-11x+15-x^2+5x+14}{(x-3)^2}$$

$$= \frac{x^2-6x+29}{(x-3)^2} = \frac{x^2-6x+3^2-9+29}{(x-3)^2} = \frac{(x-3)^2+20}{(x-3)^2}$$

$$= \frac{(x-3)^2}{(x-3)^2} + \frac{20}{(x-3)^2} = 1+\frac{20}{(x-3)^2}$$

Alternate $R(x) = x-2-\frac{20}{x-3} = x-2-20(x-3)^{-1}$

$$R'(x) = 1+20(x-3)^{-2}(1) = 1+\frac{20}{(x-3)^2} = 1+20(x-3)^{-2}$$

$R''(x) = -40(x-3)^{-3} = \frac{-40}{(x-3)^3} \overset{\text{SET}}{=} 0 \rightarrow \times$, but R'' ~~$\star$~~ @ $x=3$

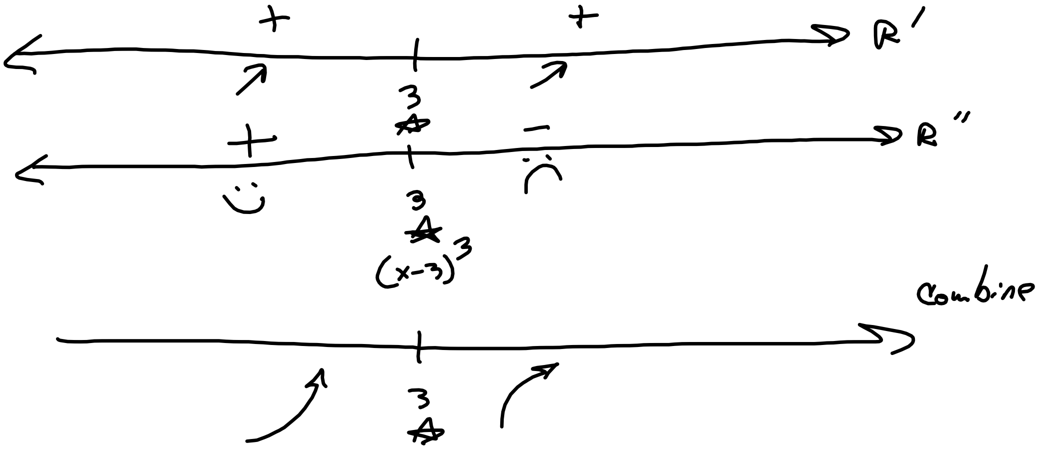
$\boxed{x=3 \; \star}$

$R'(x)=0 \Rightarrow x^2-6x+29=0 \rightarrow x^2-6x+3^2-9+29$

$= (x-3)^2+20 \overset{\text{SET}}{=} 0 \Rightarrow$

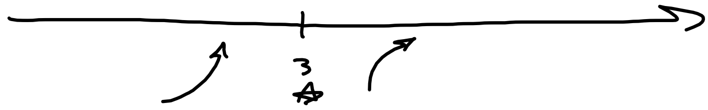
$(x-3)^2 = -20 \Rightarrow$ No real zeros

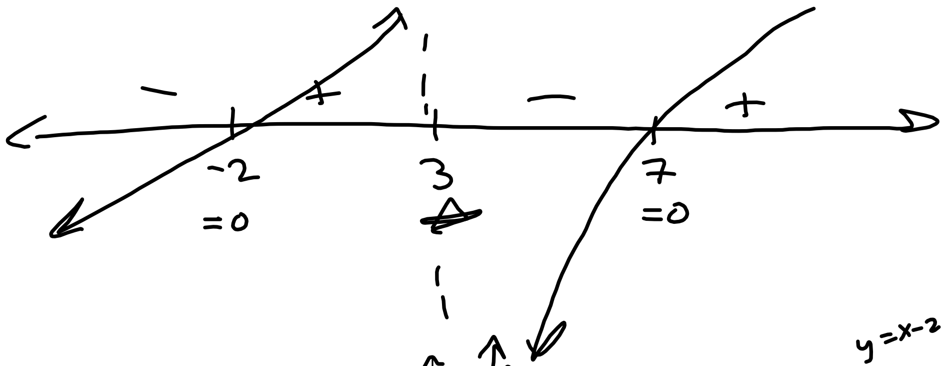
$$R'(x) = \frac{x^2-6x+29}{(x-3)^2}$$

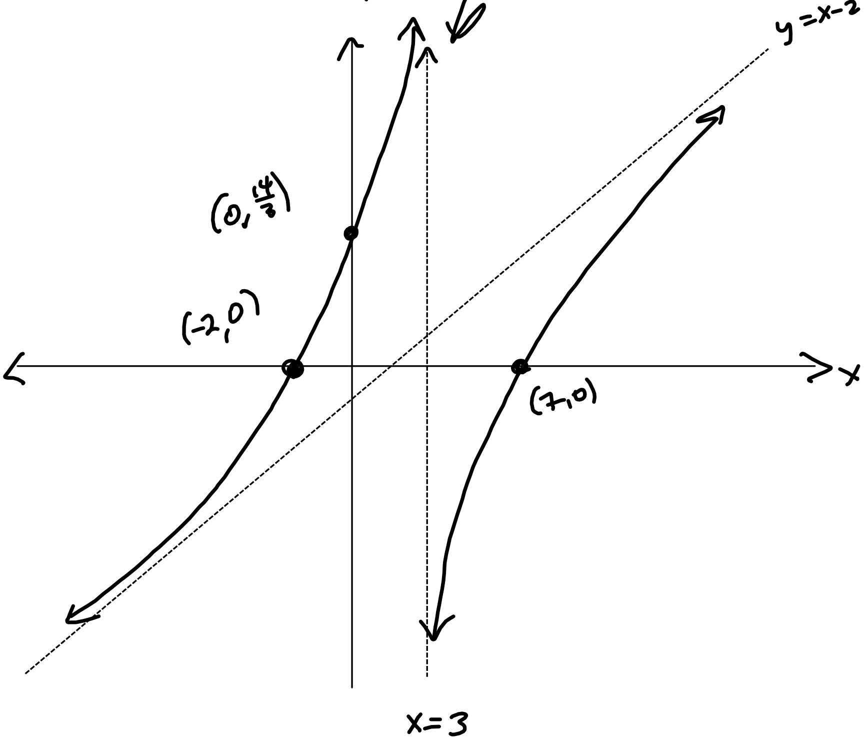


$\boxed{y=x-2 \text{ is O.A.}}$ $\boxed{\text{V.A. } x=3}$ $\boxed{\text{x-int: } (-2,0), (7,0)}$ $\boxed{\text{y-int: } (0, \tfrac{14}{3})}$

Combine R'' & R' $\dfrac{x^2-5x-14}{x-3} = R(x)$







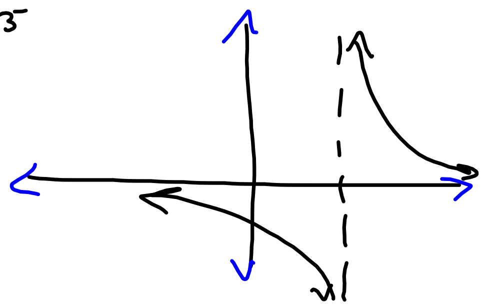
§3.4 Infinite Limits & Limits at Infinity

Vertical Asymptotes are our main intro to infinite limits

$$\lim_{x\to 3}\frac{1}{x-3} \nexists$$

$$\lim_{x\to 3^-}\frac{1}{x-3} \text{ "=" } -\infty \quad \& \quad \lim_{x\to 3^+}\frac{1}{x-3} = +\infty$$

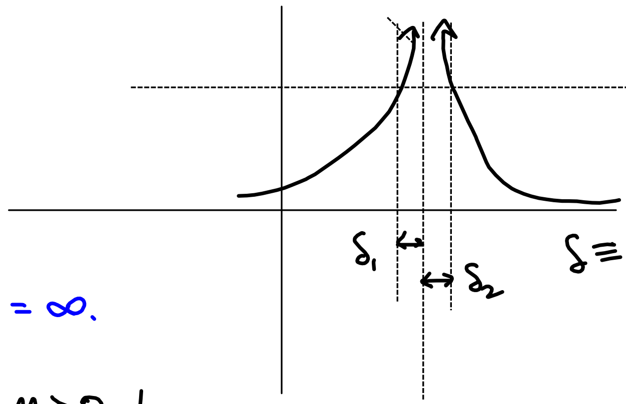
$\frac{1}{x-3} = \frac{1}{x}$ delayed by 3 units.
↳ shifted right



$x=3$

Formally,

$\lim_{x\to c} f(x) = \infty$ means, given $M>0$, $\exists\ \delta>0$ $\ni$ $|x-c|<\delta \Rightarrow f(x)>M$



$\delta \equiv \min\{\delta_1, \delta_2\}$

Claim: $\lim_{x\to 0}\frac{1}{x^2} = \infty.$

Proof Let $M>0$ be given.

Scratch:

want $\frac{1}{x^2} > M$

$x^2 < \frac{1}{M}$

$\sqrt{x^2} = |x| < \sqrt{\frac{1}{M}} \Rightarrow -\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}}.$

Proof Define $\delta = \frac{1}{\sqrt{M}}$. Then $0 < |x-0| = |x| < \delta = \frac{1}{\sqrt{M}}$

$\Rightarrow x^2 < \frac{1}{M} \Rightarrow M < \frac{1}{x^2}$ $\blacksquare$

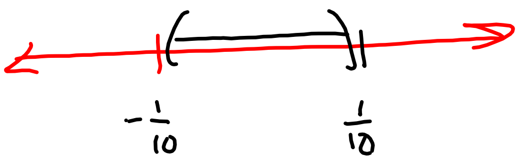
Make $\frac{1}{x^2} > 100$

$x^2 < \frac{1}{100}$

$\sqrt{x^2} = |x| < \frac{1}{10} \Rightarrow -\frac{1}{10} < x < \frac{1}{10}$

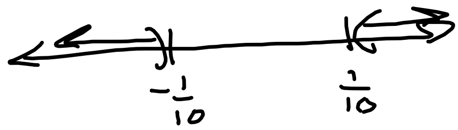
$|x| < \frac{1}{10} \Rightarrow$

$x < \frac{1}{10}$ AND $x > -\frac{1}{10}$



$|x| > \frac{1}{10}$

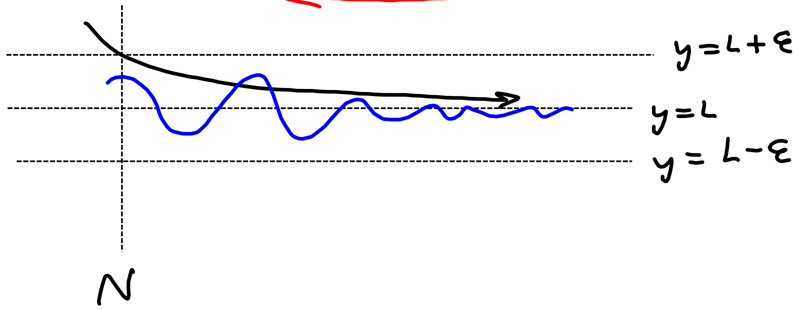
$x > \frac{1}{10}$ OR $x < -\frac{1}{10}$



Limits @ Infinity

$\lim_{x\to\infty} f(x) = L$ means given $\varepsilon>0$, $\exists\, N>0 \ni$

$|f(x)-L| < \varepsilon$ $\forall\, x > N.$ → "eventually."



Find $\lim_{x\to\infty}\left(\sqrt{x^2+7} - x\right)$

Technique: $\left(\dfrac{\sqrt{x^2+7}-x}{1}\right)\left(\dfrac{\sqrt{x^2+7}+x}{\sqrt{x^2+7}+x}\right)$

$$= \frac{x^2+7-x^2}{x+\sqrt{x^2+7}} = \frac{7}{x+\sqrt{x^2+7}} \xrightarrow{x\to\infty} 0$$

If not sure, do this:

$$x+\sqrt{x^2\left(1+\frac{7}{x^2}\right)} = x+\sqrt{x^2}\sqrt{1+\frac{7}{x^2}}$$

$$= x+|x|\sqrt{1+\frac{7}{x^2}} = x+x\sqrt{1+\frac{7}{x^2}} = x\left(1+\sqrt{1+\frac{7}{x^2}}\right)$$

$x\to\infty \rightarrow$
$x>0$

So $f(x) = \dfrac{7}{x\left(1+\sqrt{1+\frac{7}{x^2}}\right)} \xrightarrow{x\to\infty} \dfrac{7}{2x} \xrightarrow{x\to\infty} 0$

$\frac{7}{x^2} \to 0$

$$\boxed{\lim_{x\to\infty}\frac{1}{x^r} = 0 \;\; \forall\, r>0}$$

FACT

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$$

$$\left(\sqrt{9x^2+x}-3x\right)\left(\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\right)=\frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x}$$

$$=\frac{x}{\sqrt{9x^2+x}+3x}=\frac{x}{|x|\sqrt{9+\frac{1}{x}}+3x}$$

$$=\frac{x}{x\sqrt{9+\frac{1}{x}}+3x}=\frac{x}{x\left(\sqrt{9+\frac{1}{x}}+3\right)}=\frac{1}{\sqrt{9+\frac{1}{x}}+3}$$

$$\xrightarrow{x\to\infty}\frac{1}{\sqrt{9+0}+3}=\frac{1}{3+3}=\frac{1}{6}$$

When $x\longrightarrow-\infty$, $|x|=-x$

PROPER RATIONAL FUNCTION

$\dfrac{ax^n+\text{smaller}}{bx^m+\text{smaller}}$ Proper means $m>n$ H.A.: $y=0$

$\dfrac{2x^2+7x}{11x^3-5}$ ... $y=0$

Improper: $m\le n$

when $m=n$, H.A. $y=\dfrac{a}{b}$

$\dfrac{2x^2+7x}{11x^2-5}$

$y=\dfrac{2}{11}$

when $m<n$ $\dfrac{2x^2+x-15}{x+1}$

Oblique Asymptote found by division