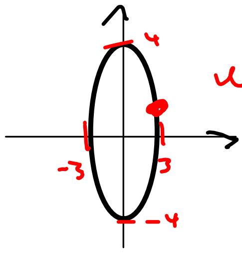
Today: Implicit Differentiation in Section 2.6 and Rates of Change in the Natural Sciences.

The ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$



Want slope @ $(\sqrt{5}, \frac{8}{3})$

OLD WAY Solve for y, explicitly:

$$\frac{y^2}{16} = 1 - \frac{x^2}{9}$$

Look @ slope @ $x=\sqrt{5}$, $y=\frac{8}{3}$

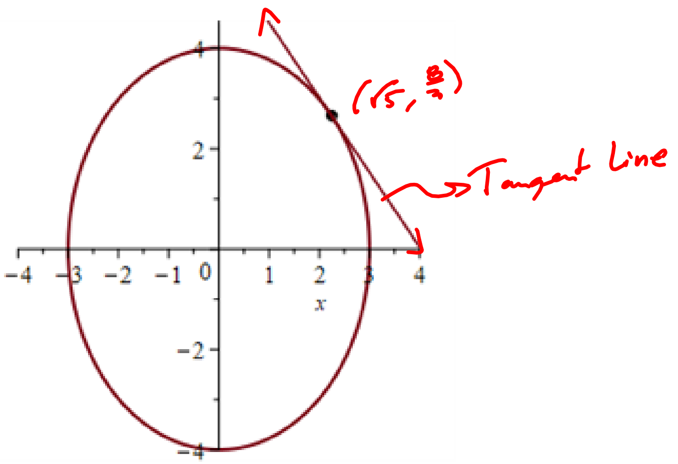
$$y^2 = 16\left(1 - \frac{x^2}{9}\right)$$

$$y = \pm\sqrt{16\left(\frac{9-x^2}{9}\right)} = \pm\frac{4}{3}\sqrt{9-x^2} = \pm\frac{4}{3}(9-x^2)^{\frac{1}{2}}$$

$f(x) = \frac{4}{3}\sqrt{9-x^2}$ is top half of the ellipse. $= \frac{4}{3}(9-x^2)^{\frac{1}{2}}$

$$f'(x) = \frac{2}{3}(9-x^2)^{-\frac{1}{2}}(-2x) = \frac{-4x}{3\sqrt{9-x^2}}$$

$$\Rightarrow f'(\sqrt{5}) = \frac{-4\sqrt{5}}{3\sqrt{9-5}} = \frac{-4\sqrt{5}}{3\sqrt{4}} = \frac{-4\sqrt{5}}{6} = \boxed{-\frac{2\sqrt{5}}{3} = \text{slope}}$$



2.6 Approach Recall Chain Rule:

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg}\cdot\frac{dg}{dx} = \cos(x^2)\cdot 2x = 2x\cos(x)$$

$f = \sin(x^2) = f(g(x)) \Rightarrow \frac{df}{dg} = \cos(g) = \cos(x^2) = \frac{d\sin(x^2)}{d(x^2)}$

$g = x^2 = g(x) \Rightarrow \frac{dg}{dx} = 2x$

$$\frac{d}{dx}[\sin(xr)] = \cos(xr)\cdot\frac{d}{dx}[xr]$$

Is r a function of x?

No: $\frac{d}{dx}[xr] = r$

Yes: $\frac{d}{dx}\left[r + x\frac{dr}{dx}\right]$ ← THIS is what we're doing today!

We assume the r is "implicitly" a function of x.

$$\frac{d}{dx}[\sin(xr)] = (\cos(xr))\underbrace{\left(r + x\frac{dr}{dx}\right)}$$

product rule on xr, treating them BOTH as functions

Usually we'll shorten things up and just write r' instead of $\frac{dr}{dx}$

or y' " " $\frac{dy}{dx}$

$$\frac{d}{dx}[y^2] = 2y\cdot\frac{dy}{dx} = 2yy'$$

$f(g(x))$:

$f = y^2 = f(y)$

$g(x) = y = y(x)$

$$\frac{d}{dx}[x^2y^3] = 2xy^3 + x^2\cdot 3y^2\cdot\frac{dy}{dx} = 2xy^3 + 3x^2y^2y'$$

$(fg)' = f'g + fg'$

Let's use this idea on the 1st example:

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\frac{d}{dx}\left[\frac{1}{9}x^2 + \frac{1}{16}y^2 = 1\right] \longrightarrow$$

$$\frac{2}{9}x + \frac{2}{16}y \cdot y' = 0 \longrightarrow$$

$$\frac{1}{8}yy' = -\frac{2}{9}x$$

Find slope @ $(\sqrt{5}, \frac{8}{3})$

$$y' = \frac{-16x}{9y}$$

$$= \frac{-16\sqrt{5}}{9\left(\frac{8}{3}\right)} = \frac{-16\sqrt{5}}{24} = \boxed{-\frac{2\sqrt{5}}{3} = m_{tan}}$$

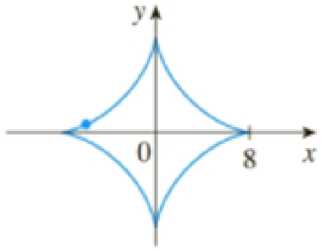
Tangent Line:

$$y = m(x - x_1) + y_1$$

$$y = -\frac{2\sqrt{5}}{3}(x - \sqrt{5}) + \frac{8}{3}$$ = tan. line to ellipse @ $(\sqrt{5}, \frac{8}{3})$

30. $x^{2/3} + y^{2/3} = 4, \quad (-3\sqrt{3}, 1)$ (astroid)

Find eq'n of tangent line at the given point.



$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$$

$$\frac{2}{3}y^{-\frac{1}{3}}y' = -\frac{2}{3}x^{-\frac{1}{3}}$$

$$y' = \frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\frac{1}{(-3\sqrt{3})^{1/3}} = \frac{1}{3^{1/3}3^{1/6}} = \boxed{\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}}$$

$(x,y) = (-3\sqrt{3}, 1)$ $\quad$ $(x,y) = (-3\sqrt{3}, 1)$

Slope.

$$(3\sqrt{3})^{\frac{1}{3}} = \left(3 \cdot 3^{\frac{1}{2}}\right)^{\frac{1}{3}} = 3^{\frac{1}{3}} \cdot 3^{\frac{1}{6}} = 3^{\frac{2}{6}}3^{\frac{1}{6}} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}$$

So, tangent line is

$$y = \frac{1}{\sqrt{3}}(x + 3\sqrt{3}) + 1 = \frac{1}{\sqrt{3}}x + 3 + 1 = \frac{1}{\sqrt{3}}x + 4$$

OR

$$y = \frac{\sqrt{3}}{3}(x + 3\sqrt{3}) + 1$$

$$\underline{y = \frac{\sqrt{3}}{3}x + 4}$$

WebAssign / Book

Have you found a CAS app for your computer/phone? Have you explored wolframalpha.com?

$\cos(xy) = 1 + \sin(y)$ Find y'

$\Rightarrow (-\sin(xy))(\underline{y + xy'}) = (\cos(y))y'$

$\frac{d}{dx}[xy] = 1y + x\frac{dy}{dx} = y + xy'$

$\Rightarrow -y\sin(xy) - xy'\sin(xy) = y'\cos(y)$

$-xy'\sin(xy) - y'\cos(y) = y\sin(xy)$

$y'(-x\sin(xy) - \cos(y)) = y\sin(xy)$

$$\boxed{y' = \frac{y\sin(xy)}{-x\sin(xy) - \cos(y)}} = \frac{-y\sin(xy)}{x\sin(xy) + \cos(y)}$$

Victoria recommends this channel on YouTube:

https://www.youtube.com/channel/UCYO_jab_esuFRV4b17AJtAw

Other students over the years: Khan Academy.

#9 $\frac{x^2}{x+y} = y^2+1$ Find $\frac{dy}{dx} = y'$

Differentiate

$$\Rightarrow \frac{2x(x+y) - x^2(1+y')}{(x+y)^2} = 2yy'$$

Expand

$$\Rightarrow 2x^2+2xy - x^2 - x^2y' = 2yy'(x+y)^2$$

Collect y' terms

$$\Rightarrow -2yy'(x+y)^2 - 2yy' = -2x^2 - 2xy + x^2$$

Factor out y' on left

$$\Rightarrow y'\left(-2y(x+y)^2 - 2y\right) = -2x^2 - 2xy + x^2$$

Divide by coefficient of y'

$$\Rightarrow y' = \frac{-2x^2 - 2xy + x^2}{-2y(x+y)^2 - 2y}$$

$$\frac{d\,f(g(x))}{d\,g(x)}$$

$$\frac{d}{dx}\left[\sin^5(x)\right] = \frac{d}{dx}\left[(\sin(x))^5\right]$$

$f(g(x)) = g(x)^5$ $\qquad \dfrac{d\,f(g(x)^5)}{d\,g(x)} \cdot \dfrac{d\,g(x)}{dx}$

$g(x) = \sin(x)$

$$= 5g(x)^4 \cdot \frac{dg}{dx} = 5(\sin(x))^4 \cdot \cos(x)$$

$$\frac{d\,(\sin(x))^5}{d\,(\sin(x))} = 5(\sin(x))^4 = 5\sin^4(x)$$

Differentiating wrt "$\sin(x)$."

§2.7 The derivative of position wrt time is velocity
" " " velocity " " " acceleration.
" " " acceleration " " " Jerk!

s = position

$$\frac{ds}{dt} = v$$

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = a$$

FALLING BODY

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0$$

Ball thrown upward from a height of 10 m @ 30 m/s.

Then $s(t) = \frac{1}{2}(9.8)t^2 + 30t + 10$

$a = 9.8$ m/s²

$$\frac{ds}{dt} = 9.8t + 30$$

$$\frac{d^2s}{dt^2} = 9.8 = a$$

$$\sqrt{3x + \sqrt{3x + \sqrt{3x}}}$$

$$= \left(3x + \left(3x + (3x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = y \Rightarrow$$

$$\frac{dy}{dx} = \frac{1}{2}\left(3x + \left(3x + (3x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(3 + \frac{1}{2}\left(3x + (3x)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(3 + \left(\frac{1}{2}\right)(3x)^{-\frac{1}{2}}(3)\right)\right)$$

$$= \frac{.5}{\left(3x + \left(3x + (3x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$