Today: Some 1.5 and 1.6. MAYbe a first epsilon-delta proof from 1.7.

We need to nudge the schedule back by a day or two, due to Labor Day and Convocation. I'd still like us to make progress Tueesday thru Thursday, pretty much as usual.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$

$$\lim_{x\to 3}\left(4x^5-3x^4+2x^3-11x^2+12x-1\right)$$

$$= 4\left(\lim_{x\to 3}(x)\right)^5 - 3\left(\lim_{x\to 3} x\right)^4 \text{ etc. ugh!}$$

Look, you mainly want to be able to plug in the values. For continuous functions,

$$\lim_{x\to 3} f(x) = f(3)$$

$f(3)$ for our example.

$$\begin{array}{r|rrrrrr} 3 & 4 & -3 & 2 & -11 & 12 & -1 \\ & & 12 & 27 & 87 & 228 & 720 \\ \hline & 4 & 9 & 29 & 76 & 240 & \boxed{719 = f(3)} \end{array}$$

Remainder Theorem. MAT 121

$$\lim_{x\to c}(\text{Poly}(x)) = \text{Poly}(c)$$

$$\lim_{x\to c}(\text{Ratl}(x)) = \text{Ratl}(c) \quad \text{if Ratl}(c) \text{ exists}$$

$R(x)$ & $P(x)$ polynomials.

Then $Q(x) = \dfrac{R(x)}{P(x)}$ is a rational function

$$\lim_{x\to c} Q(x) = Q(c) \text{ as long as } P(c) \neq 0.$$

§1.6 #16 or 17

17. 0/3 points

A graphing calculator is recommended.

(a) Estimate the value of $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{1+5x}-1}$ by graphing the function $f(x) = \dfrac{x}{\left(\sqrt{1+5x}-1\right)}$.

See video for technology solution.

If all you have is a scientific calculator, it's possible.

Doing it analytically: $(a-b)(a+b) = a^2 - b^2$

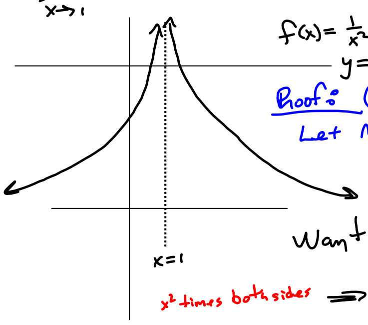
$$\left(\frac{x}{\sqrt{1+5x}-1}\right)\left(\frac{\sqrt{1+5x}+1}{\sqrt{1+5x}+1}\right) = \frac{x\left(\sqrt{5x+1}+1\right)}{(1+5x)-1} \xrightarrow{x \to 0} \text{Not Yet!}$$

$$= \frac{x\left(\sqrt{5x+1}+1\right)}{5x} = \frac{\sqrt{5x+1}+1}{5} \xrightarrow{x \to 0} \frac{\sqrt{1}+1}{5} = \frac{2}{5}$$

$$= 0.4.$$

Victoria asks about limits that approach infinity (vertical asymptotes), which are to be distinguished from limits @ infinity (horizontal asymptotes).

$\lim_{x\to 1} f(x) = \infty$ (vertical asymptote)



$f(x) = \frac{1}{x^2}$, e.g.

$|x+7|$ is how far x is from -7

$|x-(-7)|$

Proof: (Scratch)

Let $M>0$ be given

want $\delta > 0$ so that $|x-0|<\delta$ means

Want $\frac{1}{x^2} > M$

$\delta - 7$ is the difference between δ & 7

x^2 times both sides $\Rightarrow 1 > Mx^2$

$Mx^2 < 1$

$x^2 < \frac{1}{M}$

$|x| = \sqrt{x^2} < \sqrt{\frac{1}{M}}$

ϕ $|x| < \sqrt{\frac{1}{M}}$ means

$x < \sqrt{\frac{1}{M}}$ AND $x > -\sqrt{\frac{1}{M}}$

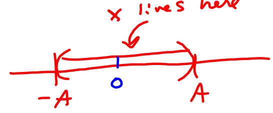
Algebra Review

Textbooks teach it this way:

$-\sqrt{\frac{1}{M}} < x < \sqrt{\frac{1}{M}}$ is an "AND" sorta OK.

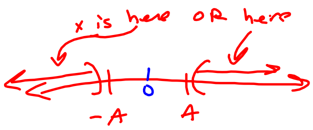
$-\sqrt{\frac{1}{M}} > x > \sqrt{\frac{1}{M}}$ BAD says $-\sqrt{\frac{1}{M}} > +\sqrt{\frac{1}{M}}$

$|x| < A$ means



$x < A$ AND $x > -A$

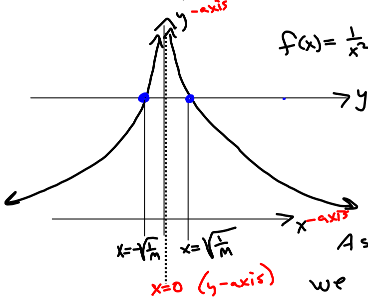
$|x| > A$ means



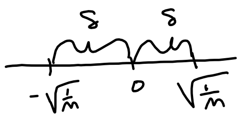
$x > A$ OR $x < -A$

Continuing:

$x < \sqrt{\frac{1}{M}}$ AND $x > -\sqrt{\frac{1}{M}}$



$f(x) = \frac{1}{x^2}$, e.g.



As long as $|x-0| < \delta = \sqrt{\frac{1}{M}}$, we have $\frac{1}{x^2} > M$.

FORMAL Proof:

Claim: $\lim_{x\to 0} \frac{1}{x^2} = \infty$

Proof: Let $M>0$ be given. Define $\delta = \sqrt{\frac{1}{M}}$.

Then if $0 < |x-0| < \delta$, we have

$x^2 < \sqrt{\frac{1}{M}}^{\,2} = \frac{1}{M} \Rightarrow$

$M < \frac{1}{x^2}$ $\blacksquare$

Limits AT infinity (Horizontal Asymptotes)

$$\lim_{x\to\infty} f(x) = L$$

$$\lim_{x\to-\infty} f(x) = L$$

same degree

$$f(x) = \frac{3x^2-5}{7x^2+\pi x-11}$$

$$\lim_{x\to\infty} f(x) = \frac{3}{7}$$

$$\lim_{x\to-\infty} f(x) = \frac{3}{7}$$

One way of working it out:

$$\frac{3x^2-5}{7x^2+\pi x-11} = \frac{x^2\left(3-\frac{5}{x^2}\right)}{x^2\left(7+\frac{11}{x}-\frac{11}{x^2}\right)}$$

$\frac{5}{x^2} \to 0$, $\frac{11}{x} \to 0$, $\frac{11}{x^2} \to 0$

$$\xrightarrow{x\to\infty} \frac{3-0}{7+0-0}$$

"End Behavior"

Prove $\lim_{x \to 3} (2x-5) = 1$ $\qquad$ $m = 2$. Let $\delta = \frac{1}{2}\varepsilon$

Proof:

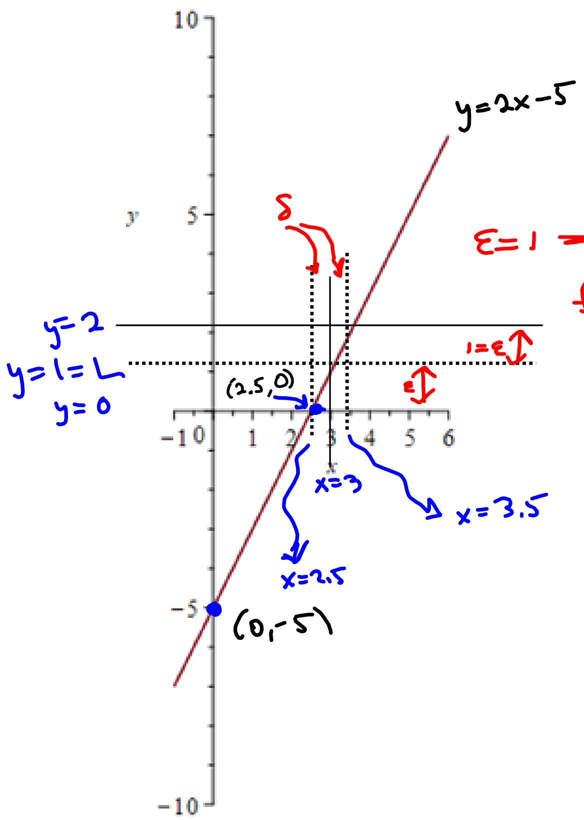
Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{2}$. Then if $0 < |x-3| < \delta$, we have

$$|(2x-5)-1| = |2x-5-1| = |2x-6|$$

$$= 2|x-3| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

$\delta = \frac{\varepsilon}{2}$ from the fact that y grows twice as fast as x, when $y = 2x-5$.

Keeping x within $\frac{\varepsilon}{2}$ units from 3 keeps $y = 2x-5$ within ε units from $y = 1$



$\varepsilon = 1 \Longrightarrow \delta = \frac{\varepsilon}{2} = \frac{1}{2}$ for $f(x) = 2x-5$

$(a^3-b^3) = (a-b)(a^2+ab+b^2)$ Perfect cubes,

$a^3+b^3 = (a+b)(a^2-ab+b^2)$ Sum & difference.

$$\lim_{x\to 3} \frac{x^3-27}{x^2+4x-21} = \lim_{x\to 3} \frac{x^3-3^3}{x^2+4x-21}$$

$$(x-3)(x+7) = x^2+4x-21$$

$$\frac{x^3\ 27}{x^2+4x-21} = \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}(x+7)} \xrightarrow{x\to 3} \frac{3^2+9+9}{10} = \boxed{\frac{27}{10}}$$