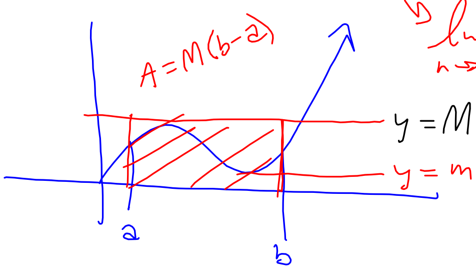
Result from §4.2

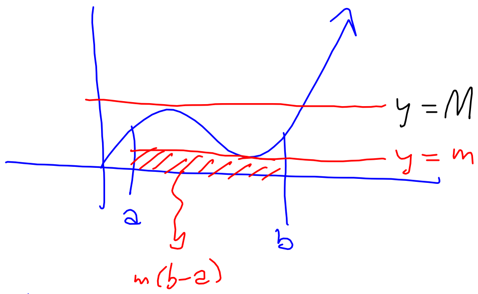
If $m = \min\limits_{x\in[a,b]} \{f(x)\}$

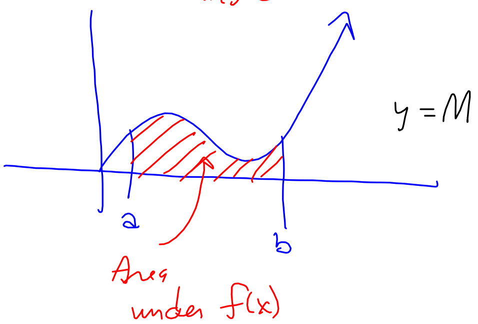
$M = \max\limits_{x\in[a,b]} \{f(x)\}$

Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k)\underbrace{\Delta x}_{\frac{b-a}{n}}$

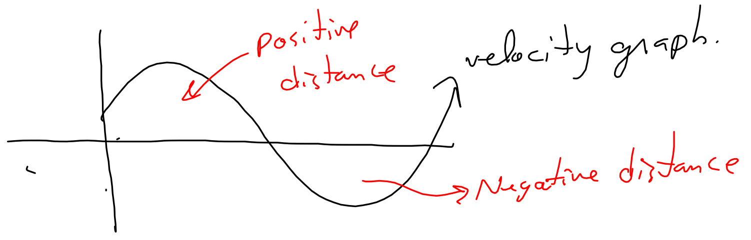




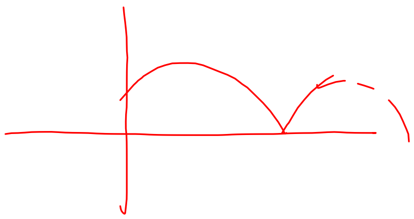


By our definition, $\int_a^b f(x)\,dx$ = Net Distance

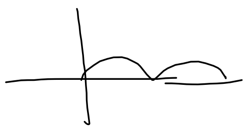
represents SIGNED area



$$\int_a^b |f(x)|\,dx = \text{GROSS DISTANCE.}$$

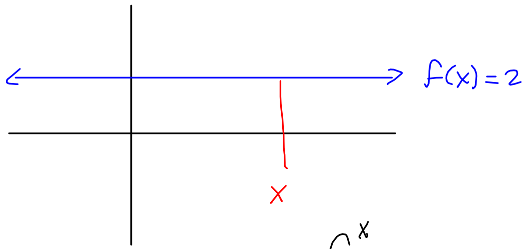


$f(x) = |\sin x|$



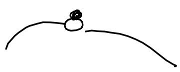
$$\int_a^x f(t)\,dt$$

$$f(x) = 2 \longrightarrow \int_0^x f(t)\,dt = \int_0^x 2\,dt$$



Then $\int_0^x f(t)\,dt$ is a function of x, say

$$g(x) = \int_0^x f(t)\,dt$$



1st Fundamental Theorem of Calculus:

FTC I:

$$\lim_{x \to c} f(x) = f(c)$$

If f is cont^{s} on $[a,b]$ and $x \in (a,b)$,

then $\int_a^x f(t)\,dt = g(x) \Longrightarrow$

$$g'(x) = f(x)$$

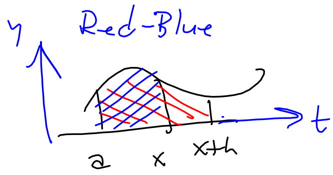
The derivative of the integral is the integrand.

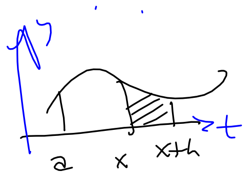
Proof

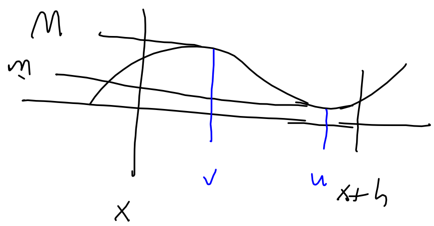
$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\left[g(x+h)-g(x)\right]$$

$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]$$







$$m(h) \le \int_x^{x+h} f(t)\,dt \le M(x+h-x) = Mh$$

By EVT $m = f(u)$ for some $u \in [x, x+h]$

and $M = f(v)$ " " v " "

$$f(u)(h) \le \int_x^{x+h} f(t)\,dt \le f(v)\,h$$

$$f(u) \le \underbrace{\frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]}_{\downarrow\, h\to 0} \le f(v)$$

$h \to 0$

$$f(x) \le g'(x) \le f(x)$$

$u, v \in [x, x+h]$ & $h \to 0 \Rightarrow$

$u \to v \to x$

$$g(x) = \int_0^x \sin t \, dt$$

$\Rightarrow g'(x) = \sin x$

b/c $\sin x$ is cnt^s on its domain $= \mathbb{R}$

$$g(x) = \int_0^x \tan t \, dt$$ can cause

you problems if $x \geq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}, \ldots$

CHAIN RULE

$$\frac{d}{dx}\left[\int_0^{3x^4} \sin(t)\, dt\right] = \left[\sin(3x^4)\right](12x^3)$$

$$\frac{d}{dx}\left[g(u(x))\right] = \frac{dg}{du} \cdot \frac{du}{dx}$$

$$= g'(u(x))\, u'(x)$$

$$\frac{d}{dx}\left[\sin x\right] = (\cos x)\left(\frac{d}{dx}(x)\right)$$

$$\frac{d}{dx}\left[\sin(3x^3 + 7x^2 - 5)\right]$$

$$= \left(\cos(3x^3 + 7x^2 - 5)\right)(9x^2 + 14x)$$

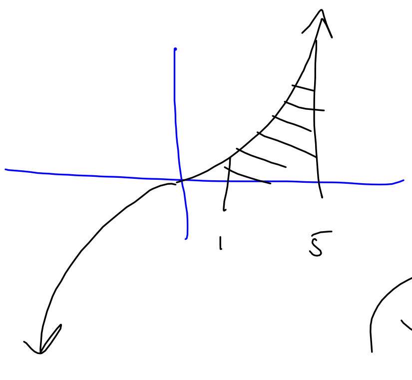
FTC II

If $f(x)$ is cont's on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$. (Always choose the $F(x)$ with $C = 0$)

$$\int \sin x \, dx = -\cos x + C$$

$$\int_1^5 x^3 \, dx = \left[\frac{1}{4}x^4\right]_1^5 = \frac{1}{4}\left[5^4\right] - \frac{1}{4}\left[5^1\right]$$

$$= \frac{1}{4}\left[625 - 5\right] = \frac{620}{4} = \frac{310}{2} = \boxed{155}$$



1
1 1
1 2 1
1 3 3 1

$$\int_1^5 x^3\,dx$$

$$\frac{b-a}{n} = \frac{4}{n} = \Delta x$$

$a=1$, $x_k = a + k\cdot\frac{4}{n} = 1 + \frac{4k}{n}$ $\qquad 1+\frac{4}{n},\ 1+\frac{8}{n}$

$$f(x_k) = f\left(1+\frac{4k}{n}\right) = \left(\frac{4k}{n}+1\right)^3$$

$$= \left(\frac{4k}{n}\right)^3 + 3\left(\frac{4k}{n}\right)^2(1) + 3\left(\frac{4k}{n}\right)^1(1)^2 + 1^3$$

combinations - Binomial Coefficients

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\binom{3}{1} = \frac{3!}{(3-1)!\,1!} = \frac{3\cdot2\cdot1}{(2\cdot1)(1)} = 3$$

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{array}$$

$$\text{Area} \approx \Delta x \sum_{k=1}^{n} f(x_k)$$

$$= \frac{4}{n}\sum_{k=1}^{n}\left[\left(\frac{4k}{n}\right)^3 + 3\left(\frac{4k}{n}\right)^2 + 3\left(\frac{4k}{n}\right) + 1\right]$$

$$\frac{4}{n}\left[\sum_{k=1}^{n}\left(\frac{4^3k^3}{n^3} + 3\left(\frac{4^2k^2}{n^2}\right) + \frac{12k}{n} + 1\right)\right]$$

$$\frac{4}{n}\left[\frac{64}{n^3}\sum k^3 + \frac{48}{n^2}\sum k^2 + \frac{12}{n}\sum k + \sum 1\right]$$

$$\frac{4\cdot 64}{n^4}\sum k^3 + \frac{4\cdot48}{n^3}\sum k^2 + \frac{4\cdot12}{n^2}\sum k + \frac{4}{n}\sum 1$$

$$\frac{256}{n^4}\left[\frac{n^4+\sim\sim}{4}\right] + \frac{192}{n^3}\left[\frac{n^3+\sim\sim}{3}\right] + \frac{48}{n^2}\left[\frac{n^2+\sim\sim}{2}\right] + \frac{4}{n}[n]$$

$$= \frac{256}{n^4}\cdot\frac{n^4}{4} + \frac{256}{n^4}\left[\frac{\sim\sim}{4}\right] + \cdots$$

$\xrightarrow{n\to\infty}$ 0, b/c highest degree upstairs is 3, so $\frac{\sum n^3}{n^4} = \frac{\sum}{n} \xrightarrow{n\to\infty} 0$

$$\xrightarrow{n\to\infty} \frac{256}{4} + \frac{192}{3} + \frac{48}{2} + 4$$

$$= 64 + 64 + 24 + 4$$

$= 128 + 28 = 156$ is off by 1.

But pretty darn good.