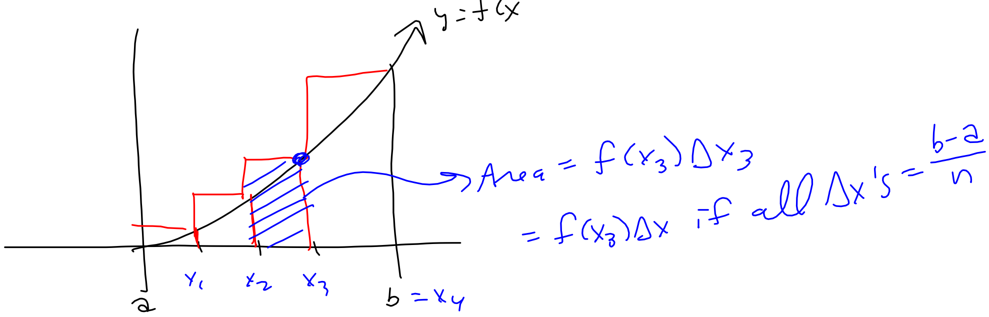
§4.1 Areas under a curve

§4.2 Getting Close to Definite Integral



Estimating the area with rectangles

Even-width rectangles.

width $= \frac{b-a}{n} = \frac{b-a}{4} = \Delta x$ = step size.

"Mesh of the partition" = width of widest subinterval. (Eventually $\longrightarrow 0$)

Right-endpoint Riemann Sum.

Area$_1$ + Area$_2$ + ... + Area$_n$

$= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$

$= \sum_{k=1}^{n} f(x_k)\Delta x_k \approx$ Area $= \sum_{k=1}^{n} f(x_k)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k)$

$\sum_{k=1}^{n} k = 1+2+3+\cdots+n$

$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$\sum_{k=1}^{4} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta y$

$= \Delta x\,(f(x_1) + f(x_2) + f(x_3) + f(x_4))$

$= \Delta x \sum_{k=1}^{4} f(x_k)$

Lower ↓ σ, upper ↓ Σ

Leading up to $\int f(x)\,dx$ — "S" stands for sum.

$= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\left(\frac{b-a}{n}\right)$

Estimate the area under $f(x)=x^2$ on $[0,1]$ with 4 rectangles.

$n=4,\ a=0,\ b=1 \Rightarrow \frac{b-a}{n}=\Delta x=\frac{1-0}{4}=\frac{1}{4}$

$x_1=a+\Delta x=0+\frac{1}{4}$

$x_2=x_1+\Delta x=a+2\Delta x=0+\frac{2}{4}=\frac{2}{4}=\frac{1}{2}$

$x_3=\frac{3}{4},\ x_4=1=b$

$$\text{Area}\approx\sum_{k=1}^{n} f(x_k)\Delta x=\left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right)+\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right)+\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)+\left(1^2\right)\left(\tfrac{1}{4}\right)$$

$$=\Delta x\sum f(x_k)$$

$$=\frac{1}{4}\left[\left(\tfrac{1}{4}\right)^2+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{3}{4}\right)^2+1^2\right]$$

$$=\frac{1}{4}\left[\frac{1}{16}+\frac{4}{16}+\frac{9}{16}+\frac{16}{16}\right]$$

$$=\left(\frac{1}{\cancel{4}_{2}}\right)\left(\frac{1}{16}\right)\left[\cancel{30}_{15}\right]=\frac{1}{32}\cdot 15=\frac{15}{32}$$

More rectangles $\Rightarrow$ Better Precision

$n\to\infty \Rightarrow$ Calculus! $\Rightarrow$ EXACT area.

---

§4.2 Let's take $n\to\infty$!

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}=\frac{n^2+n}{2}=\frac{n^2+\text{smaller degree}}{2}=\frac{n^2+\sim}{2}$$

$1+2+3+\cdots+99+100$

$$\left(\frac{100}{2}\right)(101)=\frac{100(101)}{2}=\frac{n(n+1)}{2}$$

Prove that $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

Proof by induction

$\sum_{k=1}^{1}k=1=\frac{1(1+1)}{2}=1$ ✓ $\Rightarrow 1\in S'=\{n \mid \text{The statement holds.}\}$

We want to show $S'=\mathbb{N}=\{1,2,3,\dots\}$.

Suppose $n\in S'$ we show $n+1\in S'$.

$n\in S' \Rightarrow \sum_{k=1}^{n}k=\frac{n(n+1)}{2}$ Induction Hypothesis.

$$\Rightarrow \sum_{k=1}^{n+1}k=\underbrace{1+2+3+\cdots+(n-1)+n}+(n+1)$$

$$=\frac{n(n+1)}{2}+(n+1)$$

$$=\frac{n^2+n}{2}+\frac{2(n+1)}{2}=\frac{n^2+n+2n+2}{2}$$

$$=\frac{n^2+3n+2}{2}=\frac{(n+1)(n+2)}{2}=\frac{(n+1)((n+1)+1)}{2}$$

$\Rightarrow n+1\in S'$ & so true $\forall n\in\mathbb{N}$, by Principle of Mathematical Induction

Prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

FOR 1 BONUS ASSIGNMENT.

Keep our eye on what matters:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{(n^2+n)(2n+1)}{6} = \frac{2n^3+n^2+2n^2+n}{6}$$

$$= \frac{2n^3+\sim}{6} = \frac{2n^3}{6} + \frac{\sim}{6} = \frac{n^3}{3} + \sim$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{(n^2+n)^2}{4} = \frac{n^4+2n^3+n^2}{4} = \frac{n^4+\sim}{4}$$

Summarize

$$\sum_{k=1}^{n} k = \frac{n^2+\sim}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n^3+\sim}{3},$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4+\sim}{4}$$

Recall $\int x\,dx = \frac{x^2}{2}+C$, $\int x^2\,dx = \frac{x^3}{3}+C$, $\int x^3\,dx = \frac{x^4}{4}+C$

§4.2 Main Skill

Find the EXACT area under

$f(x) = x^2$ on $[0, 1]$.

$a = 0,\ b = 1$

$\frac{b-a}{n} = \frac{1}{n} = \Delta x$

Find expression for $x_k = a + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$

$$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{1}{n} \sum_{k=1}^{n} x_k^2 = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 = \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \frac{1}{n} \cdot \frac{1}{n^2} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n^3}{3} + \frac{an^2 + bn + c}{3}\right] = \frac{\cancel{n^3}}{3\cancel{n^3}} + \frac{an^2 + bn + c}{3n^3} \xrightarrow{n \to \infty} \boxed{\frac{1}{3} = \text{Area}}$$

$\frac{an^2 + bn + c}{3n^3} \to 0$

This is the ~ stuff

Always $\to 0$!

Area under $f(x) = x^3$ on $[0,5]$

$a=0, b=5,$

$\Delta x = \frac{5}{n}$

$x_k = a + k\Delta x = 0 + \frac{5k}{n}$

$$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{5}{n} \sum_{k=1}^{n} \left(\frac{5k}{n}\right)^3 = \frac{5}{n} \sum_{k=1}^{n} \frac{5^3 k^3}{n^3}$$

$$= \frac{5}{n} \cdot \frac{5^3}{n^3} \sum_{k=1}^{n} k^3 = \frac{5^4}{n^4} \cdot \frac{n^4 + \cdots}{4} \xrightarrow{n \to \infty} \frac{5^4}{4}$$

$$\underbrace{\int_0^5 x^3\,dx = \frac{x^4}{4}\Big]_0^5 = \frac{5^4}{4} - \frac{0^4}{4} = \frac{5^4}{4}}$$

Fundamental Theorem of Calculus II.

FTC II