Today! Prove Product

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Proof

$$f(x)g(x) = h(x) \Rightarrow \frac{h(x+h) - h(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h}$$

$$= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}$$

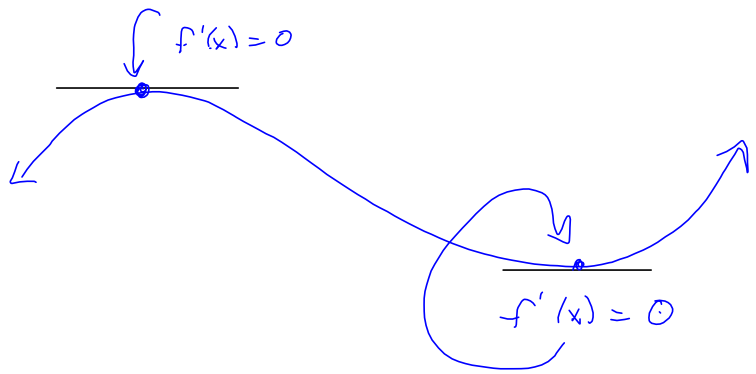
$$\xrightarrow{h \to 0} f'(x)g(x) + f(x)g'(x)$$

$$h(x) = x^2 \sin(x) \Rightarrow f(x) = x^2, \; g(x) = \sin(x)$$

$$h'(x) = \underbrace{2x}_{f'}\underbrace{\sin(x)}_{g} + \underbrace{x^2}_{f}\underbrace{\cos(x)}_{g'}$$

THE BIG APP: optimization

"Take $f'(x)$, set $= 0$, solve."



$\boxed{E}$ $f(x) = x(x-2)(x+3)$

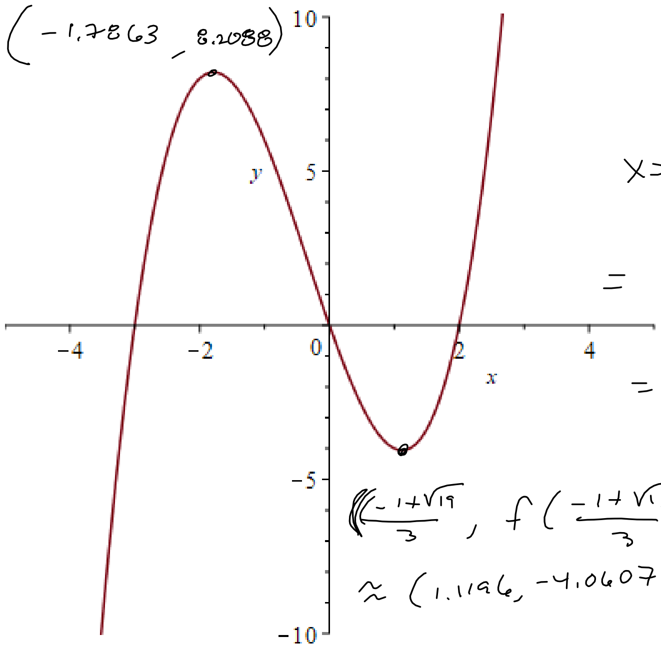
$f'(x) = 3x^2 + 2x - 6 = 0 \Rightarrow a = 3, b = 2, c = -6$

$\Rightarrow b^2 - 4ac = 2^2 - 4(3)(-6)$

$= 4 + 72$

$= 76$

$(-1.7863, 8.2088)$



$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2 \pm 2\sqrt{19}}{6}$

$= \dfrac{-1 \pm \sqrt{19}}{3}$

$2 \lfloor 76$
$2 \lfloor 38$
19

$\sqrt{76}$
$= \sqrt{4 \cdot 19}$
$= 2\sqrt{19}$

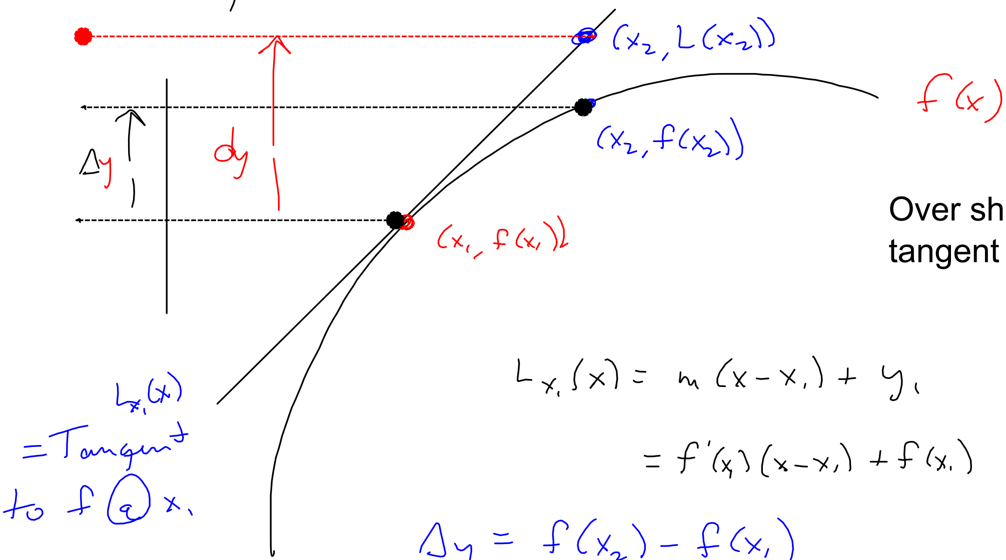
$\left(\dfrac{-1+\sqrt{19}}{3}, f\left(\dfrac{-1+\sqrt{19}}{3}\right)\right)$

$\approx (1.1196, -4.0607)$

-4.060672588

See applications coming up!

Tangent Line Stuff $\Longleftrightarrow$ Differentials



Over short distances, the tangent line is close to f(x).

$$L_{x_1}(x) = m(x - x_1) + y_1$$
$$= f'(x_1)(x - x_1) + f(x_1)$$

$$\Delta y = f(x_2) - f(x_1)$$

$$dy =$$

$$L(x_2) = f'(x_1)(x_2 - x_1) + f(x_1)$$

$$\underbrace{L(x_2) - f(x_1)}_{dy} = f'(x_1)(x_2 - x_1)$$

$$dy = f'(x_1)(x_2 - x_1) = f'(x_1)\Delta x \equiv f'(x_1)\,dx$$

Application: How much paint to cover a sphere of radius $r = 1$, if the coat is .5cm thick?

$\boxed{\text{M1 :}}$ $y = Vol = V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

Vol of paint $= V(1.5) - V(1)$

$\frac{3}{2} = 1.5$

$= \frac{4}{3}\pi(1.5)^3 - \frac{4}{3}\pi(1)^3$

$\left(\frac{3}{2}\right)^3 = \frac{27}{8} = 1.5^3$

$= \frac{4}{3}\pi\left[\frac{27}{8} - 1\right] = \frac{4}{3}\pi\left[\frac{19}{8}\right] = \frac{19\pi}{6}\ cm^3$

Differential Approximation

$\Delta y \approx dy = f'(x)\,dx = V'(r)\,dr = V'(r)\Delta r$

$r_1 = 1$

$r_2 = 1.5$

$\Delta r = .5$

$= \left(4\pi(1)^2\right)(.5) = \boxed{2\pi\ cm^3}$

Bad estimate when $\Delta r \rightarrow$ B.G.