Today §1.7 Precise Def'n of Limit

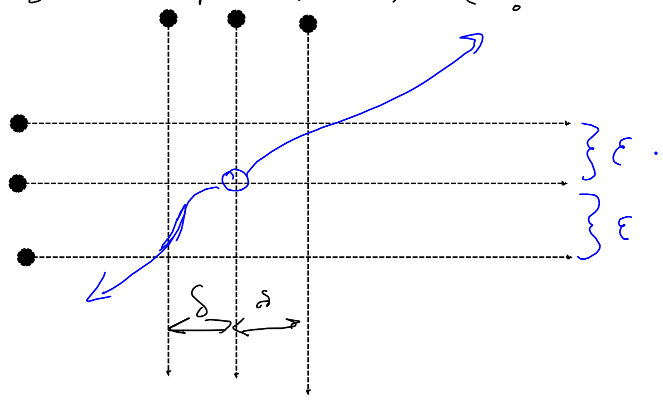
Formally:

$\lim_{x \to c} f(x) = L$ means

$\forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$0 < |x-c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.



$x \to c$

Linear Proofs will be required

Quadratic (and higher-degree) will be bonus.

$\boxed{E}$ Claim: $\lim_{x \to 3} 5x - 7 = 8$

Scratch: want $|5x-7-8| < \varepsilon$

$\Rightarrow |5x-15| = 5|x-3| < 5\delta \equiv \varepsilon$

$\Rightarrow \delta = \frac{\varepsilon}{5}$

Proof: Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{5}$.

Then $0 < |x-3| < \delta \Rightarrow |5x-7-8|$

$= |5x-15| = 5|x-3| < 5\delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ∎

Claim: $\lim_{x \to 7} 3x - 2 = 19$

Proof Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{3}$. Then

$0 < |x-7| < \delta \Rightarrow |f(x) - 19| = |3x-2-19|$

$= |3x-21| = 3|x-7| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ ∎

Advanced Calculus.

Real Analysis.

Quadratic Limits (Bonus)

Claim: $\lim_{x\to 2} x^2-2x+3 = 3$

Scratch: Want $|x^2-2x+3-3| < \varepsilon$

$\Rightarrow |x^2-2x| = |x(x-2)| = |x||x-2|$

$\boxed{< |x|\delta \le 3\delta \text{ is the key, from scratch.}}$

$3\delta \le \varepsilon \Rightarrow \delta \le \frac{\varepsilon}{3}$

Need a ceiling for $|x|$ here.

Assume $\delta \le 1$.

Then $|x| \dots$

$\delta \le 1$ means $0 < \boxed{|x-2| < \delta \le 1}$

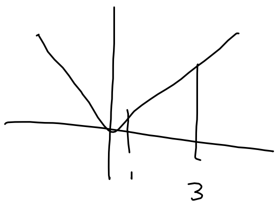
$|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

So what's this say about $|x|$?

It says $|x| < 3$!



Proof: Let $\varepsilon > 0$ be given.

Define $\delta = \min\{1, \frac{\varepsilon}{3}\}$. Then $0 < |x-2| < \delta$

$\Rightarrow |x^2-2x+3-3| = |x^2-2x| = |x||x-2|$

$< |x|\delta \le 3\delta \le 3\cdot\frac{\varepsilon}{3} = \varepsilon$ ∎

Claim

$$\lim_{x\to 5} x^2-2 = 23$$

Scratch: $|x^2-2-23| = |x^2-25| = |x+5|\underbrace{|x-5|}_{<\delta}$

Need a bound on $|x+5|$

Assume $\delta \le 1$. Since $x \to 5$, that means

$|x-5| < \delta \le 1$

$|x-5| \le 1$

$-1 \le x-5 \le 1$

$4 \le x \le 6$

$9 \le x+5 \le 11$

So $|x+5| \le 11$!

$\delta \le 2$

$|x-5| \le 2$

$-2 \le x-5 \le 2$

$3 \le x \le 7$

$8 \le x \le 12$

$\frac{\varepsilon}{12}$

So define $\delta = \min\{1, \frac{\varepsilon}{11}\}$!

Proof:

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{11}\}$.

Then $0 < |x-5| < \delta \Longrightarrow$

$$|x^2-2-23| = |x^2-25| = |x+5||x-5|$$

$$< |x+5|\,\delta \le 11\,\delta \le 11\cdot\frac{\varepsilon}{11} = \varepsilon \quad \blacksquare$$

$f(x)$ is continuous at $x=c$ means

$$\lim_{x\to c} f(x) = f(c)$$