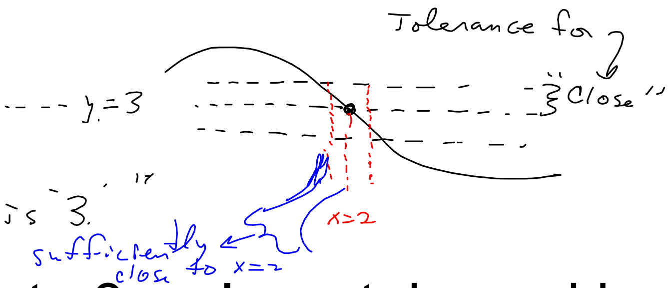
§1.5 Limits

$\lim\limits_{x \to 2} f(x) = 3$

"Limit, as x approaches 2 is 3."



I can make *f* as close to 3 as I want, by making x sufficiently close to 2.

" x is close to 2"

How far you are from $x=2$:

$|x-2|$

"Very close" means "Distance away is small."

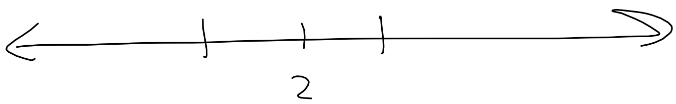
Let δ be a very small number.

" x is close to 2":

$|x-2| < \delta$

Tell me how close you want $f(x)$ to be to $y=3$.

$|f(x)-3| <$ small



Challenge: Want $|f(x)-3| <$ small $= \varepsilon$

Response: Make $|x-2| <$ small $= \delta$

TASK: To find a sufficiently small δ that works for a GIVEN ε.

I want $5x-7$ to be within $.2$ units of $y=3$ in the vicinity of $x=2$.

Make $|(5x-7)-3| < .2$

by making $|x-2|$ small enough.

Scratch:

Want $|5x-7-3| < .2 \Longrightarrow$

$\Longleftarrow |5x-10| < .2 \Longrightarrow$

Proof

Scratch

$\Longleftarrow 5|x-2| < .2 \Longrightarrow$

$\Longleftarrow |x-2| < \frac{.2}{5} = .04$

This says, $|f(x)-3| < .2$ when $|x-2| < .04$

Claim: $\lim_{x \to 2} (5x-7) = 3$

WORKS FOR ANY Given tolerance

Proof: Let $\varepsilon > 0$ be given. Then define $\delta = \frac{\varepsilon}{5}$,

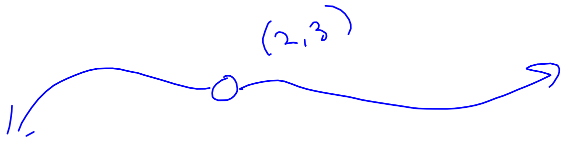
and so $|x-2| < \delta \implies$

$$|f(x)-3| = |5x-7-3| = |5x-10| = 5|x-2| < 5\delta$$

$$= 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$\varepsilon = .2$ is just one specific case, but the math is all the same.

$\lim_{x \to 2} f(x) = 3$ is a 2-sided idea

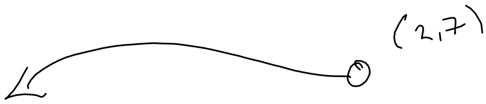


Note the open dot @ $x = 2$, $y = 3$.

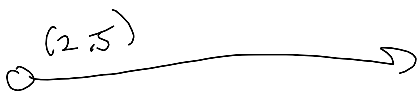
"HOLE"

$\lim_{x \to 2^-} f(x) = 7$ and $\lim_{x \to 2^+} f(x) = 5$

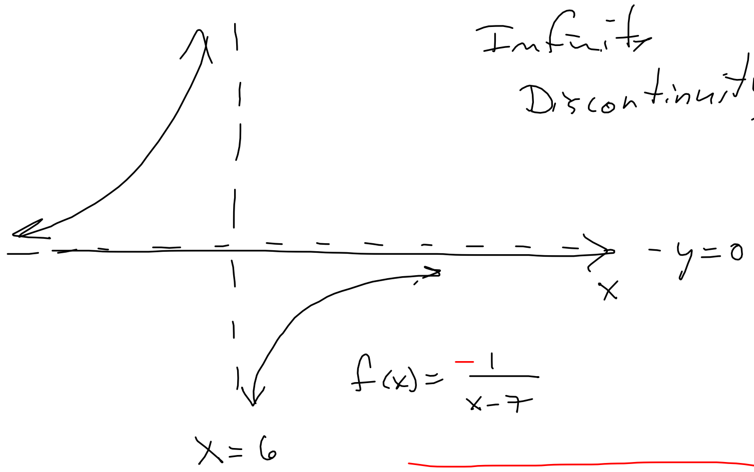
For this, $\lim_{x \to 2} f(x) \nexists$



"Jump Discontinuity"



Infinity Discontinuity



$f(x) = \frac{-1}{x-7}$

Typically we say $\boxed{\lim_{x \to 7^-} f(x) = \infty \text{ and } \lim_{x \to 7^+} f(x) = -\infty}$

A bit improper, but useful.

But if we're just trying to see if it approaches a real number, we'd say $\lim_{x \to 7} f(x) \nexists$.

How to show $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$

Give me a BIG number. I show that

I can make $f(x) > BIG$ by making $|x-2| < \delta$

Let $M = 80$

want $\left| \frac{1}{x-2} \right| > M = 80$

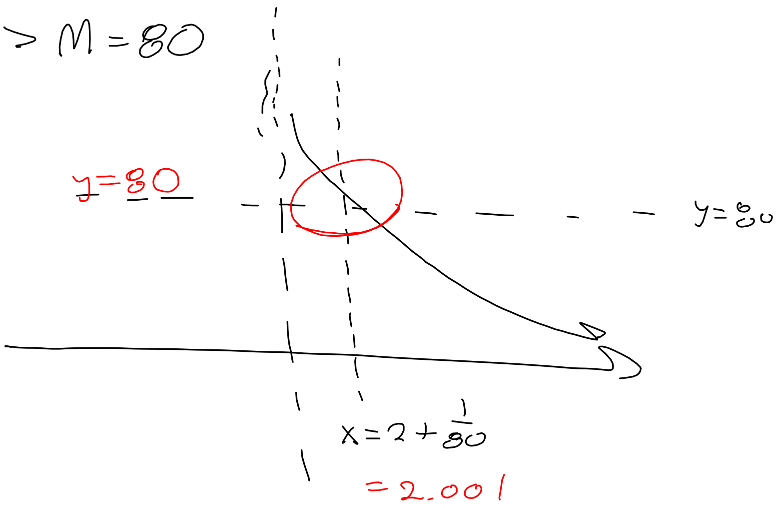
$\frac{1}{|x-2|} = \left| \frac{1}{x-2} \right| > 80$ 1000

$1 > 80|x-2|$ 1000

$80|x-2| < 1$ 1000

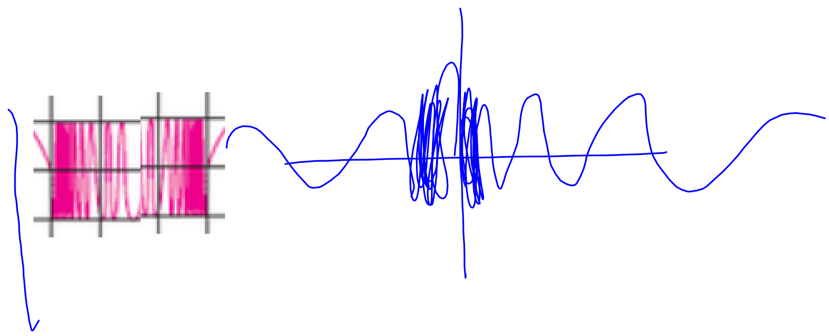
$|x-2| < \frac{1}{80} \equiv \delta$

$\frac{1}{1000} = .001$



Pathologies to test our understanding:

$f(x) = \sin\left(\frac{1}{x}\right)$



$x \sin\left(\frac{1}{x}\right) \xrightarrow{x \to 0} 0$