MAT 201 Final Test – Spring, 2015 Name KEY

200 Points Comprehensive

Instructor: Dr. Harry S. Mills

Show all work. Do your own work. Submit problems in the proper order. Spread your work out! If you get stuck, **start a fresh piece of paper**. You can always *insert* more pages if you do it this way. Only your *name* should be on this cover sheet. Test is 1 hour, 50 minutes. Start a 12:10. End at 2:00.

1. Let $f(x)=2x^2-3$. Find $\dfrac{df}{dx}$ in two ways:
   a. (10 pts) the limit definition. b. (5 pts) the easy way.
2. Let $f(x)=2x^2-3$.
   a. (5 pts) Find an equation of the tangent line to f at $x=2$.
   b. (5 pts) Sketch a graph of f and the tangent line you obtained in part a.
   c. (5 pts) Use your tangent line to approximate $f(2.5)$.
3. Evaluate the following limits.
   a. (5 pts) $\lim\limits_{x\to 3}\left(\dfrac{2x^2-11x+15}{3x^2-7x-6}\right)$ b. (5 pts) $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-11x+15}{3x^2-7x-6}\right)$ c. (5 pts) $\lim\limits_{x\to 3^-}\left(\dfrac{|x-3|}{x^2-x-6}\right)$
4. (5 pts) Prove that $\lim\limits_{x\to 3}(2x-5)=1$.
5. (5 pts) Convince me – without *solving* – that $f(x)=x^3-x^2-16x+16$ has a zero in the interval $(0,2)$. I suggest use of a major theorem.
6. Sketch the graph of $f(x)=x^3-x^2-16x+16$, showing all extremes and inflection points. Be smart about the time spent on calculations (a lot) versus points available for doing so (very little).
   a. (5 pts) x-values corresponding to max/min. (Corresponding y-value: 0 points)
   b. (5 pts) x-values corresponding to inflection points. (Corresponding y-value: 0 points)
   c. (5 pts) Sign pattern on $f'(x)$ and $f''(x)$.
   d. (5 pts) x-intercepts and y-intercept.
   e. (5 pts) Sketch, showing extremes, inflection point, and "shape" (concavity).
7. Find $\dfrac{dy}{dx}$:
   a. (5 pts) $y=-\dfrac{1}{\sqrt[5]{x^2}}+5x^2-4$
   b. (5 pts) $y=2x^3\cos(x^2-3)$

201 FINAL SPRING '15

① $f(x) = 2x^2-3 \Longrightarrow$

ⓐ $\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^2-3-(2x^2-3)}{h}$

$= \frac{2(x^2+2xh+h^2)\cancel{-3}-2x^2\cancel{+3}}{h}$

$= \frac{2x^2+4xh+2h^2-2x^2}{h} = \frac{4xh+2h^2}{h} = 4x+2h$

$h \to 0 \Longrightarrow \boxed{4x = f'(x)}$ 10 pts

ⓑ $\boxed{f'(x) = 4x}$ 5 pts

③ $f(x) = 2x^2-3 \Longrightarrow$

ⓐ $f'(2) = 4(2) = 8 = m$

$x_1 = 2 \Longrightarrow f(x_1) = 2(2)^2-3$

$= 8-3 = 5 = y_1$

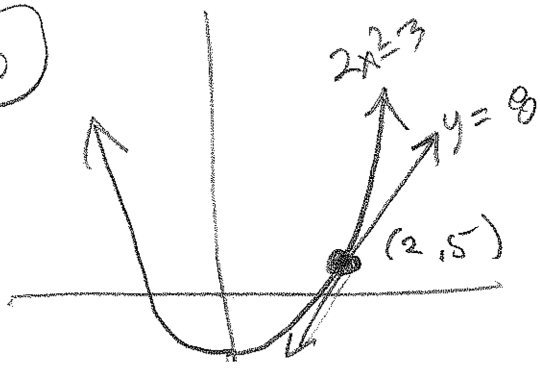
$y = m(x-x_1)+y_1$

$\boxed{y = 8(x-2)+5}$ 5 pts

$= 8x-16+5$

$y = 8x-11$

ⓑ



5 pts

$f(2.5) = f(\frac{5}{2}) = 2(\frac{5}{2})^2-3$

$= 2(\frac{25}{4})-3 = \frac{25}{2}-\frac{6}{2} = \frac{19}{2}$

$= 9.5 = f(2.5)$

ⓒ $f(2.5) \approx 8(2.5-2)+5$

$= 8(.5)+5 = 4+5 = \boxed{9 \approx f(2.5)}$ 5 pts

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③ ⓐ $\frac{2x^2-11x+15}{3x^2-7x-6} = \frac{(2x-5)(x-3)}{(3x+2)(x-3)}$

$= \frac{2x-5}{3x+2} \xrightarrow{x \to 3} \frac{2(3)-5}{3(3)+2} = \frac{6-5}{9+2} = \boxed{\frac{1}{11}}$ 5pts

$(x \neq 3)$

ⓑ $\frac{2x^2-11x+15}{3x^2-7x-6} \xrightarrow{x \to \infty} \frac{2}{3}$ 5pts

ⓒ $\lim_{x \to 3^-} \frac{|x-3|}{x^2-x-6} = \lim_{x \to 3^-} \frac{|x-3|}{x^2-x-6} = \lim_{x \to 3^-} \frac{-(x-3)}{(x-3)(x+2)}$

$= \lim_{x \to 3^-} -\frac{1}{x+2} = -\frac{1}{3+2} = \boxed{-\frac{1}{5}}$ 5pts

ⓓ $\lim_{x \to -\infty} \left(\sqrt{25x^2+3x} + 5x\right)$

$= \lim_{x \to -\infty} \frac{25x^2+3x-25x^2}{\sqrt{25x^2+3x} - 5x} = \lim_{x \to -\infty} \frac{3x}{\sqrt{25x^2}\sqrt{1+\frac{3x}{25x^2}} - 5x}$

$= \lim_{x \to -\infty} \frac{3x}{-5x\sqrt{1+\frac{3}{25x}} - 5x} = \lim_{x \to -\infty} \frac{3x}{-5x\left(\sqrt{1+\frac{3}{25x}} + 1\right)}$

$= \frac{3}{-5(1+1)} = -\frac{3}{10}$

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④ $\lim_{x \to 3} (2x-5) = 1$ 5pts

<u>PF</u> Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{2}$.

Then $0 < |x-3| < \delta \Longrightarrow$

$|(2x-5)-1| = |2x-6| = 2|x-3| < 2\delta = \varepsilon$ ■

⑤ $f(x) = x^3 - x^2 - 16x + 16 \Longrightarrow$ 5pts

$f(0) = 16$, & $f(2) = 2^3 - 2^2 - 16(2) + 16$

$= 8 - 4 - 32 + 16$

$= 4 - 16 = -12$

f is poly $\Rightarrow$ cnts on $[0,2]$, difbl on $(0,2)$,

$f(0) = 16 > 0 > -12 = f(2) \Longrightarrow$

$\exists\ c \in (0,2) \ni f(c) = 0$, by IVT.

⑥
$$\begin{array}{r|rrrr} 4 & 1 & -1 & -16 & 16 \\ & & 1 & 0 & -16 \\ \hline & 1 & 0 & -16 & 0 \end{array}$$

$x^2 - 16 = (x-4)(x+4)$

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ⓔ x-int: (1, 0), (-4, 0), (4, 0)

y-int: (0, 16) $(x-1)(x-4)(x+4)$

$f'(x) = 3x^2 - 2x - 16 \overset{SET}{=} 0$ $f''(x) = 6x - 2 \overset{SET}{=} 0$

$6x = 2$

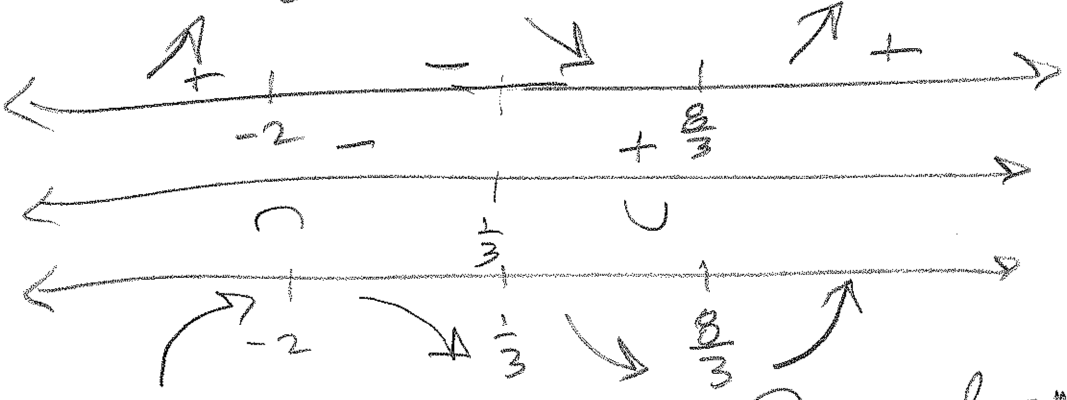
$x = \frac{2}{6} = \frac{1}{3}$

$3x^2 - 8x + 6x - 16$

$= x(3x-8) + 2(3x-8)$

$= (x+2)(3x-8)$

$x = -2, \frac{8}{3}$



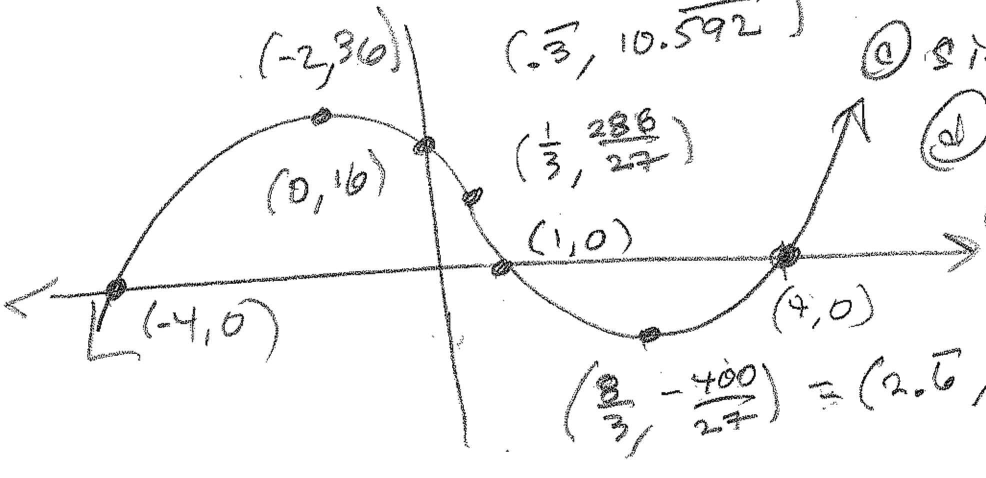
ⓐ x-vals - max/min (5pts)

ⓑ x-vals - inflection (5pts)

ⓒ sign pattern (5pts)

ⓓ x- & y-int (5pts)

ⓔ sketch (5pts)



$\left(\frac{8}{3}, -\frac{400}{27}\right) = (2.\overline{6}, -14.\overline{814})$

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⑦ ⓐ $y = -x^{-2/5} + 5x^2 - 4 \longrightarrow$

$$\boxed{y' = \frac{2}{5}x^{-7/5} + 10x}$$

5pts

ⓑ $y = 2x^3 \cos(x^2-3) \longrightarrow$ 5pts

$$\boxed{y' = 6x^2 \cos(x^2-3) - 2x^3(\sin(x^2-3))(2x)}$$

ⓒ $x^2y^2 - 3xy - 3x = 2y^2 + xy \longrightarrow$

$2xy^2 + x^2(2yy') - 3y - 3xy' - 3 = 4yy' + y + xy'$

$2x^2yy' - 3xy' - 4yy' - xy' = y - 2xy^2 + 4y + 3$

$y'(2x^2y - 3x - 4y - x) = -2xy^2 + 4y + 3$

$$\boxed{y' = \frac{-2xy^2 + 4y + 3}{2x^2y - 4x - 4y}}$$

5pts

ⓓ $y = \int_0^{x^3} \frac{t^2 \sin(t)}{10 - \cos^2(t)}\, dt \longrightarrow$

$$\boxed{y' = \left(\frac{(x^3)^2 \sin(x^3)}{10 - \cos^2(x^3)}\right)(3x^2)}$$

10pts

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⑧ $V = \frac{4}{3}\pi r^3$

(10 pts)

$dV = 4\pi r^2 dr$

$= 4\pi (10)^2 (0.1)$

$= 4\pi (100)(.1)$

$= 4\pi (10) = \boxed{40\pi \text{ cm}^3} \approx \Delta V$

125.6637061

2pts for knowing vol of sphere *sight*

(5 pts)

⑨ ⓐ f is cnts on [a,b]

f is diffbl on (a,b)

ⓑ ~~AS~~ [a,b] = [0,2]

$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{-12-16}{2} = -\frac{28}{2} = -14$

$f'(x) = 3x^2 - 2x - 16 \overset{SET}{=} -14$

(5 pts)

$\Rightarrow 3x^2 - 2x - 2 = 0$

$a = 3, b = -2, c = -2$

$b^2 - 4ac = (-2)^2 - 4(3)(-2)$

$= 4 + 24 = 28$

$\boxed{c = \frac{1 \pm \sqrt{7}}{3}}$

≈ 1.215250437

$x = \frac{2 \pm \sqrt{28}}{2(3)} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$

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⑩ $g(x) = 3x^2 - x - 7 \Longrightarrow$ Average value

g_{avg} of g on $[a,b] = [0,1]$ is

$$g_{AVG} = \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{1}{1-0}\int_0^1 (3x^2 - 2x - 7)\,dx$$

$$= \int_0^1 (3x^2 - 2x - 7)\,dx = \left[x^3 - x^2 - 7x\right]_0^1 = 1 - 1 - 7 = -7$$

10pts

Set $g(x) = -7$:

$3x^2 - 2x - 7 = -7 \Longrightarrow$

$3x^2 - 2x = 0 \Longrightarrow$

$x(3x-2) = 0 \Longrightarrow$

$x \in \{0, \frac{2}{3}\}$

$\exists \; \boxed{c = \frac{2}{3} \in (0,1)}$

If they do
MAVG instead
of gavg, as directe
but solve g(x) = mavg
correctly (c=2 OR C=
then 4pts

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⑪ $h(x) = 2\sin x \cos x + x$

[illegible]

ⓐ $h'(x) = 2\cos^2 x - 2\sin^2 x + 1$

$= 2\cos^2 x - 2(1-\cos^2 x) + 1$

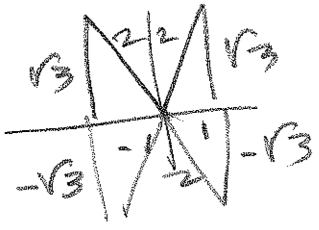
$= 2\cos^2 x - 2 + 2\cos^2 x + 1$

$= 4\cos^2 x - 1 \overset{SET}{=} 0 \longrightarrow$

$\cos^2 x = \frac{1}{4} \Longrightarrow$

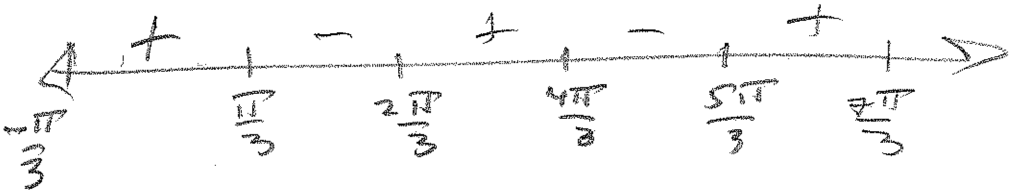
$\cos x = \pm\frac{1}{2}$

5pts



$\longrightarrow \boxed{x \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}}$

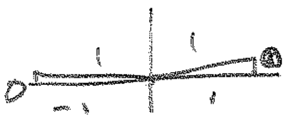
$4\cos^2 x - 1 = h'$ 1

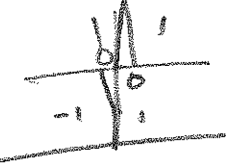


1

ⓑ $h''(x) = 8\cos x(-\sin x) = -8\sin x \cos x \overset{SET}{=} 0$ 1

$\sin x = 0$ $\cos x = 0$





5pts

$\boxed{x \in \left\{\cancel{0}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}}$

I didn't say $(0, 2\pi)$ as intended in the question

Final

Alternates

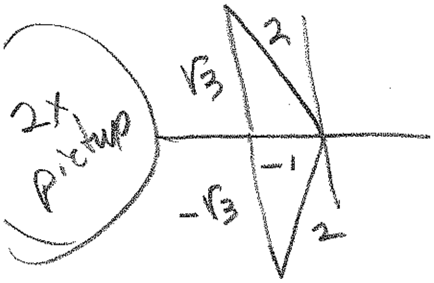
(11a) $h(x) = 2\sin x \cos x + x$

$= \sin(2x) + x \Longrightarrow$ Want $x \in (0, 2\pi)$,

so $2x \in (0, 4\pi)$

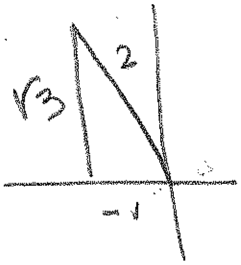
$h'(x) = 2\cos(2x) + 1 \overset{\text{SET}}{=} 0$

$\Longrightarrow \cos(2x) = -\frac{1}{2}$



$2x = \frac{2\pi}{3}$

$\boxed{x = \frac{\pi}{3}.}$



Also $2x = \frac{2\pi}{3} + 2\pi$

$\Longrightarrow x = \frac{\pi}{3} + \pi = \boxed{\frac{4\pi}{3} = x}$

$2x = \frac{4\pi}{3}$

$\boxed{x = \frac{2\pi}{3}}$

Also $2x = \frac{4\pi}{3} + 2\pi$

$\Longrightarrow x = \frac{2\pi}{3} + \pi = \boxed{\frac{5\pi}{3} = x}$

You could also get to $2\cos(2x) + 1$ via

$4\cos^2 x - 1 = 0$

$4\left(\frac{1 + \cos(2x)}{2}\right) - 1 = 0$

$2 + 2\cos(2x) - 1 = 0$

$2\cos(2x) + 1 \doteq 0$, etc.

FINAL

Alternate

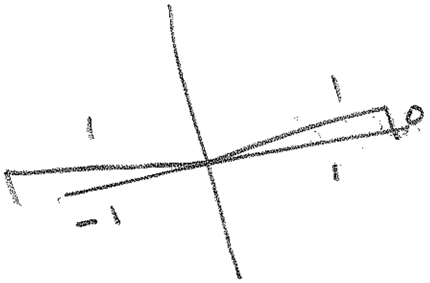
$'(x) = 2\cos(2x)+1 \implies$

$h''(x) = -4\sin(2x) \overset{SET}{=} 0$

Again, $x \in (0, 2\pi)$ means we account for all $2x \in (0, 4\pi)$

$\sin(2x) = 0$

So $2x = 0$, $2x = \pi$



Also $2x = \pi + 2\pi$

$2x = 0$

$\rightarrow \boxed{x = 0}$

$2x = \pi$

$\rightarrow \boxed{x = \frac{\pi}{2}}$

$\rightarrow x = \frac{\pi}{2} + \pi$

$\boxed{x = \frac{3\pi}{2}}$

Also $2x = 0 + 2\pi$

$\rightarrow \boxed{x = \pi}$

$x \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

I meant to say $x \in (0, 2\pi)$, but didn't, not that many got far enough to list all real solutions...

201 FINAL

10 pts

(12) (a) $\int \csc^2(x)\,dx = -\csc(x)\cot(x) + C$

(b) $\int \frac{dx}{(\sqrt{x}+1)^3}$ $u = \sqrt{x} + 1$ 10 pts

$du = \frac{1}{2}x^{-\frac{1}{2}}dx \Longrightarrow$

$dx = 2\sqrt{x}\,du$

$u=\sqrt{x}+1 \Longrightarrow u-1 = \sqrt{x}$, so

$2\sqrt{x}\,du = 2(u-1)du$, so ...

$$\int \frac{2(u-1)\,du}{u^3} = \int\left(\frac{2u}{u^3} - \frac{2}{u^3}\right)du$$

$$= \int(2u^{-2} - 2u^{-3})du = \frac{2u^{-1}}{-1} - \frac{2u^{-2}}{-2} + C$$

$$= \boxed{-2(\sqrt{x}+1)^{-1} + (\sqrt{x}+1)^{-2} + C}$$

(c) $\int x\csc^2(4x^2)\,dx = \frac{1}{8}\int \csc^2(4x^2)\cdot 8x\,dx$

$$= \frac{1}{8}(-\cot(4x^2)) + C$$

Final

FOR TESTING
CENTER
version

$$) \int 3x(4x-1)^{\frac{3}{2}}dx$$

$u = 4x-1 \Longrightarrow 4x = u+1$

$du = 4dx \Longrightarrow x = \frac{u+1}{4}$

$dx = \frac{du}{4}$

This gives $\int 3\left(\frac{u+1}{4}\right)(u)^{\frac{3}{2}}\frac{du}{4}$

$$= \frac{3}{16}\int u^{\frac{3}{2}}(u+1)du = \frac{3}{16}\int\left(u^{\frac{5}{2}}+u^{\frac{3}{2}}\right)du$$

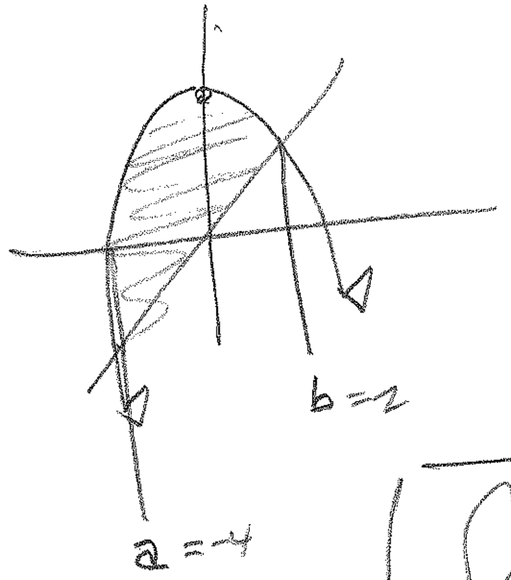
$$= \frac{3}{16}\left[\frac{2}{7}u^{\frac{7}{2}}+\frac{2}{5}u^{\frac{5}{2}}\right]+C$$

$$= \frac{3}{16}\left[\frac{2}{7}(4x-1)^{\frac{7}{2}}+\frac{2}{5}(4x-1)^{\frac{5}{2}}\right]+C$$

$$\text{OR } \frac{3}{16}\left[\frac{2}{7}\sqrt{(4x-1)^7}+\frac{2}{5}\sqrt{(4x-1)^5}\right]+C$$

201 FINAL

③ $y=2x,\ y=8-x^2$ 5pts



$$8-x^2=2x$$

$$-x^2-2x+8=0$$

$$x^2+2x-8=0$$

$$(x+4)(x-2)=0$$

$$x\in\{-4,2\}$$

$$\boxed{\int_{-4}^{2}(8-x^2-2x)\,dx}$$

$$=\int_{-4}^{2}(-x^2-2x+8)\,dx=\left[-\tfrac{1}{3}x^3-x^2+8x\right]_{-4}^{2}$$

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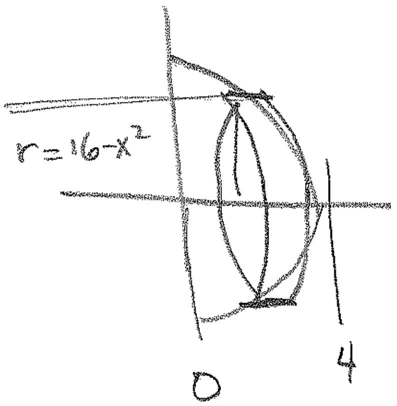
⑭ $y=16-x^2,\ x=0,\ y=0$ about x-axis

ⓐ Disc Method (10 pts)

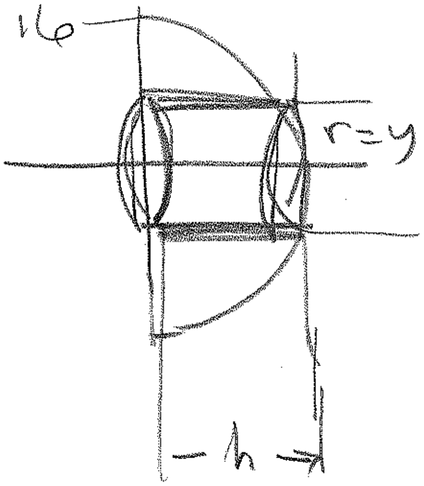


$$\boxed{\pi\int_0^4 (16-x^2)^2\,dx}$$

$$\pi r^2 \Delta x$$

ⓑ Shell Method (10 pts)

$$\boxed{2\pi\int_0^{16} y\sqrt{16-y}\ dy}$$



$y=16-x^2$

$x^2=16-y$

$x=\pm\sqrt{16-y}$

$h=\sqrt{16-y}$

$2\pi r h \Delta y$

$2\pi y\sqrt{16-y}$

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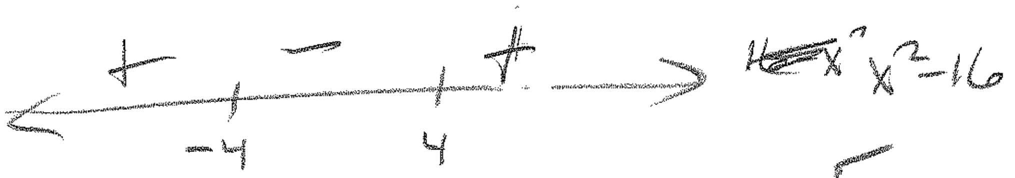
(15) (B1) $\int_0^5 |x^2-16|\,dx$ (Bonus 10pts)

$$= -\int_0^4 (x^2-16) + \int_4^5 (x^2-16)\,dx$$



$$= \left[-\left(\tfrac{1}{3}x^3-16x\right)\right]_0^4 + \left[\left(\tfrac{1}{3}x^3-16x\right)\right]_4^5$$

$$= -\left[\tfrac{1}{3}(4)^3-16(4)-0\right] + \left[\tfrac{1}{3}(5^3)-16(5)-\left(\tfrac{1}{3}(4)^3-16(4)\right)\right]$$

$$= -\left[\tfrac{64}{3}-64\right] + \left[\tfrac{125}{3}-80-\left(\tfrac{64}{3}-64\right)\right]$$

$$= -\left[-\tfrac{128}{3}\right] + \left[\tfrac{125-240}{3}-\left(-\tfrac{128}{3}\right)\right]$$

$$= \tfrac{128}{3}+\tfrac{128}{3}-\tfrac{115}{3} = \tfrac{256-115}{3} = \tfrac{141}{3} = \boxed{47}$$

46.999734

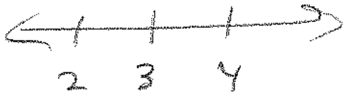
201 FINAL

⑯ $\lim_{x \to 3} (x^2-3x-10) = -10$ BONUS 10pts

$|x^2-3x-10-(-10)| = |x^2-3x| = |x||x-3| < \underline{|x|}\,\delta$

Assume $\delta \leq 1$. Then, $x \to 3 \Longrightarrow$

$2 < x < 4$, i.e., $\underline{|x| < 4}$ The key.



<u>Write the proof:</u>

Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{4}\}$. Then

$0 < |x-3| < \delta \Longrightarrow |x^2 3x-10-(-10)|$

$= |x||x-3| < |x|\,\delta < 4\delta \leq 4 \cdot \frac{\varepsilon}{4} = \varepsilon$ $\blacksquare$

201 FINAL

⑰ $\lim_{x \to -\infty} \left(\sqrt{25x^2+3x} + 5x \right)$ Bonus 10 pts

$$= \lim_{x \to -\infty} \left(\sqrt{25x^2+3x} + 5x \right) \left(\frac{\sqrt{25x^2+3x} - 5x}{\sqrt{25x^2+3x} - 5x} \right)$$

$$= \lim_{x \to -\infty} \frac{25x^2+3x-25x^2}{\sqrt{25x^2+3x} - 5x}$$

$$= \lim_{x \to -\infty} \frac{3x}{\sqrt{25x^2\left(1+\frac{3x}{25x^2}\right)} - 5x}$$

$$= \lim_{x \to -\infty} \frac{3x}{5|x|\sqrt{1+\frac{3}{25x}} - 5x}$$

$$= \lim_{x \to -\infty} \frac{3x}{-5x\sqrt{1+\frac{3}{25x}} - 5x}$$

$$= \lim_{x \to -\infty} \frac{3x}{-5x\left(\sqrt{1+\frac{3}{25x}} + 1\right)}$$

$$= \lim_{x \to -\infty} \frac{3}{-5\left(\sqrt{1+\frac{3}{25x}} + 1\right)} = \frac{3}{-5(1+1)} = -\frac{3}{10}$$