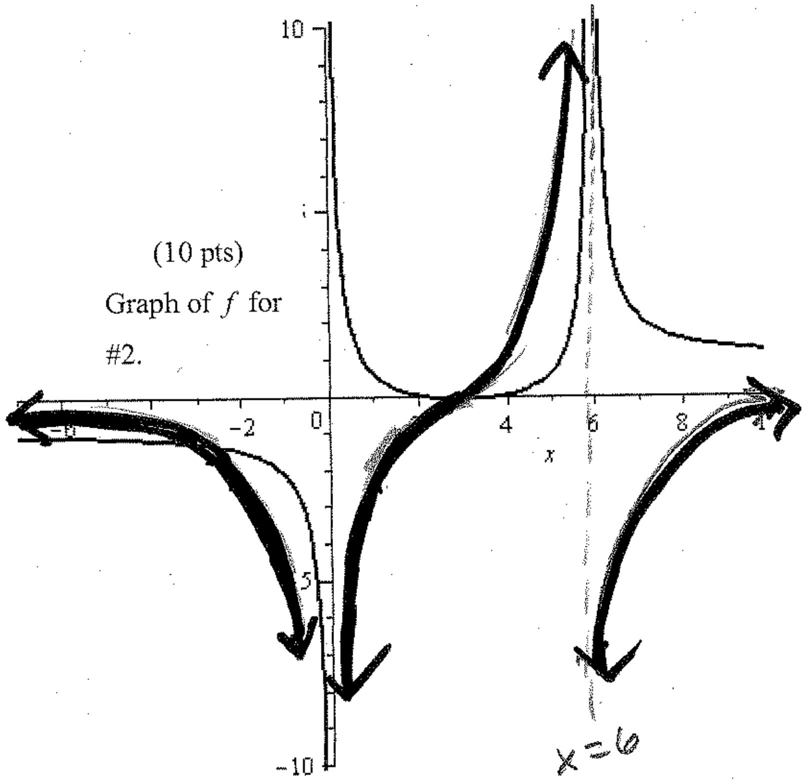
MAT 201 Test 2 – Spring, 2015 Name ~~KEY~~

100 Points Covers Chapter 2 NO GRAPHING CALCULATORS!!!

Show all work. Do your own work. Without supporting work, the slightest misstep leads to zero credit. Spread your work out! If you get stuck, start a fresh piece of paper. You can always *insert* more pages if you do it this way. No work should be on this cover sheet, except the graph for #2.

(10 pts) Graph of f for #2.



1. Let $f(x)=\sqrt{x+1}$.
   a. (5 pts) Find an equation of the tangent line to f at the point $(3,\ 2)$.
   b. (5 pts) Sketch a graph showing f and the tangent line to f at $x=3$.
2. (10 pts) The graph of a function f is given on the right. On the same set of axes, sketch a graph of f'.
3. (5 pts each) Differentiate the following with respect to the main variable.
   a. $f(x)=x^5-6x^{\frac{7}{3}}+6\sqrt[3]{x^7}+4x^{\frac{2}{5}}-\frac{3}{2}x^{-\frac{2}{3}}$
   b. $h(\omega)=\left(\omega^2+3\omega+13\right)\left(\omega^3-7\omega^2+6\omega-11\right)$
   c. $H(t)=\dfrac{t^2+3t+13}{t^3-7t^2+6t-11}$
   d. $g(x)=\left(x^2+3x+13\right)^3\left(x^3-7x^2\right)^{-5}$
   e. $r(x)=\dfrac{\left(x^2+3x+13\right)^3}{\left(x^3-7x^2\right)^5}$
   f. $Q(t)=\dfrac{\sin\left(t^2-3t\right)}{\cos^2(5t)}$
   g. $R(x)=\dfrac{\csc^3(5x)}{\tan(\pi x)}$

See Other, Revised Test At End.

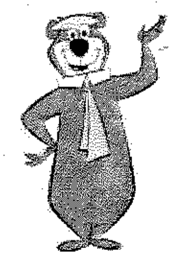
4. (10 pts) Show that $f(x)=x^3-6x^2+15x-7$ has no tangent line with a slope of $m=-2$.
5. Consider the relation $y\sin(2x)=x\cos(2y)$.
   a. (5 pts) Use implicit differentiation to find $y'=\dfrac{dy}{dx}$.
   b. (5 pts) Find an equation of the tangent line to the curve at the point $\left(\dfrac{\pi}{2},\dfrac{\pi}{4}\right)$.

6. (10 pts) A lighthouse is located on a small island *exactly* $\sqrt{3}$ km from the nearest point P on a straight shoreline. The light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km away from P?
7. (10 pts) The radius of a sphere is 3 cm, with a possible error in measurement of 0.1 cm.
   a. Use differentials to estimate the error in the volume calculated from this measurement of the radius.

      $\left(\text{Hint: The volume of a sphere is given by } V = \frac{4}{3}\pi r^3\right)$.

   b. What is the relative error?
   c. What is the percent error?

Work up to 2 Bonus questions for up to 10 points extra.



Bonus (5 pts) Show, using implicit differentiation, that any tangent line to a circle, at a point $P = (x_1, y_1)$ on the circle, is perpendicular to the radius QP, where $Q = (h, k)$ is the center of the circle.

Bonus (5 pts) Prove that $\lim_{x\to 3}(x^2 - 2x + 1) = 4$.

Bonus (5 pts) Give a rough sketch of the graph of $y = 3(2x-7)^{2/3} - 3$, by transforming the graph of a basic function.

Bonus (5 pts) Convince me that $f(x) = x^4 - 3x^3 - 22x^2 + 78x - 60$ has a zero in the interval $(4,5)$, without, you know, actually finding it.

Bonus (5 pts) Approximate $\sin(48^0)$ using the linearization.

201 TEST 2 Spring '15

1. $f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$,

(a) Tangent Line @ $(3,2)$

(b) Picture

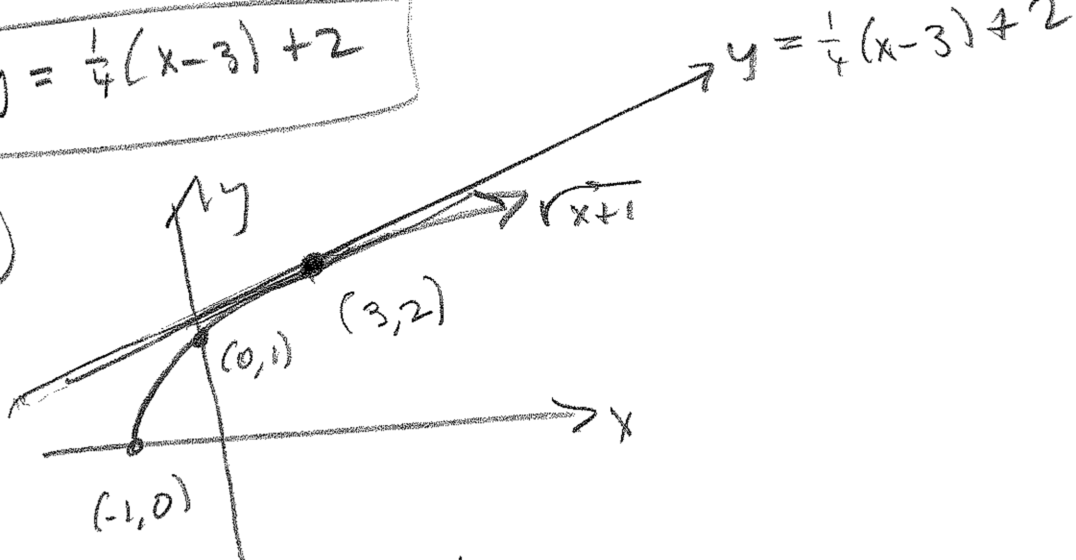
(a) $f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$$m_{tan} = f'(3) = \frac{1}{2}(3+1)^{-\frac{1}{2}}$$
$$= \frac{1}{2}(4)^{-\frac{1}{2}}$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{4}}\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = m_{tan}$$

$y = m(x - x_1) + y_1$

$$\boxed{y = \frac{1}{4}(x-3) + 2}$$

(b)



2. SEE Cover Sheet

3. (a) $f(x) = x^5 - 6x^{\frac{7}{3}} + 6x^{\frac{7}{3}} + 4x^{\frac{2}{5}} - \frac{3}{2}x^{-\frac{2}{3}}$

$$\Longrightarrow \boxed{f'(x) = 5x^4 + \frac{8}{5}x^{-\frac{3}{5}} + x^{-\frac{5}{3}}}$$

3b $h(\omega) = (\omega^2 + 3\omega + 13)(\omega^3 - 7\omega^2)$

$$\Rightarrow \boxed{h'(\omega) = (2\omega + 3)(\omega^3 - 7\omega^2) + (\omega^2 + 3\omega + 13)(3\omega^2 - 14\omega)}$$

3c $H(t) = \dfrac{t^2 + 3t}{t^3 + 6t - 1} \Rightarrow$

$$\boxed{H'(t) = \frac{(2t+3)(t^3+6t-1) - (t^2+3t)(3t^2+6)}{(t^3+6t-1)^2}}$$

3d $g(x) = (x^2+3x+13)^3(x^3-7x^2)^{-5} \Rightarrow g'(x) =$

$$\boxed{3(x^2+3x+13)^2(2x+3)(x^3-7x^2)^{-5} + (x^2+3x+13)^3(-5(x^3-7x^2)^{-6})(3x^2-14x)}$$

3e $r(x) = \dfrac{(x^2+3x+13)^3}{(x^3-7x^2)^5}$ See #3d! $r'(x) =$

$$\boxed{\frac{3(x^2+3x+13)^2(2x+3)(x^3-7x^2)^5 - (x^3+3x+13)^3(5(x^3-7x^2)^4(3x^2-14x)}{\left[(x^3-7x^2)^5\right]^2}}$$

3f $Q(t) = \dfrac{\sin(t^2-3t)}{\cos(5t)} \Rightarrow Q'(t) =$

$$\frac{(\cos(t^2-3t))(2t-3)(\cos(5t)) - (\sin(t^2-3t))(-5\sin(5t))}{\cos^2(5t)}$$

201 TEST 2

(3g) $R(x) = \dfrac{\csc^3(5x)}{\tan(\pi x)} \implies R'(x) =$

$$\boxed{\frac{\left(3\csc^2(5x)\right)\left(-5\csc(5x)\left(\cot(5x)\right)\right)\left(\tan(\pi x)\right) - \left(\csc^3(5x)\right)\left(\pi\sec^2(\pi x)\right)}{\tan^2(\pi x)}}$$

201 TEST 2

4 $f(x) = x^3 - 6x^2 + 15x - 7$ has no tan line with slope $m_{tan} = -2$.

$\boxed{Pf}$ $\mathcal{S}$ it DOES. Then

$f'(x) = 3x^2 - 12x + 15 = -2$ for some $x \in \mathbb{R}$

$\Rightarrow 3x^2 - 12x + 17 = 0$

$\Rightarrow b^2 - 4ac = (-12)^2 - 4(3)(17)$

$= 144 - 204$

$= -60 < 0 \Rightarrow$ NO REAL SOL'N!

$\Rightarrow$ No $x \ni f'(x) = m_{tan} = -2$ ∎

5 (a) $y \sin(2x) = x \cos(2y) \Rightarrow$

$y' \sin(2x) + (y)(2\cos(2x)) = \cos(2y) + x(-2\sin(2y))y'$

$y' \sin(2x) + 2y \cos(2x) = \cos(2y) - 2y' \sin(2y)$

$\Rightarrow y'(\sin(2x) + 2\sin(2y)) = \cos(2y) - 2y\cos(2x)$

$$\boxed{\Rightarrow y' = \frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2\sin(2y)}}$$

201 TEST 2 (5)

(5b) Tan Line @ $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

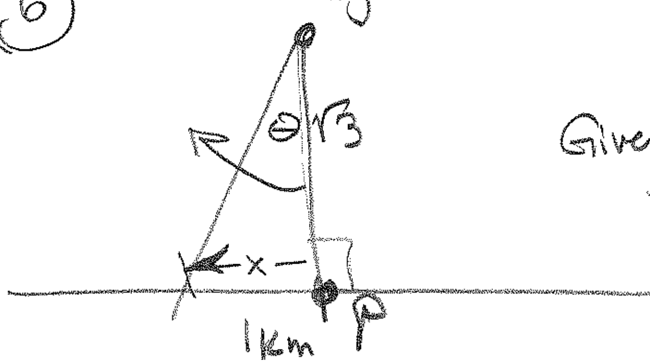
$$y'\Big|_{\left(\frac{\pi}{2},\frac{\pi}{4}\right)} = \frac{\cos\left(2\left(\frac{\pi}{4}\right)\right) - 2\left(\frac{\pi}{4}\right)\cos\left(2\left(\frac{\pi}{2}\right)\right)}{\sin\left(2\left(\frac{\pi}{2}\right)\right) + 2\sin\left(2\left(\frac{\pi}{4}\right)\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\cos(\pi)}{\sin(\pi) + 2\sin\left(\frac{\pi}{2}\right)} = \frac{0 - \frac{\pi}{2}(-1)}{0 + 2(1)} = \frac{\pi}{4} = m$$

$\longrightarrow y = m(x - x_1) + y_1$

$$\boxed{y = \frac{\pi}{4}\left(x - \frac{\pi}{2}\right) + \frac{\pi}{4}}$$

(6) Lighthouse want $\left.\frac{dx}{dt}\right|_{x=1}$



Given $\frac{5\text{ rev}}{\text{min}} \cdot \frac{2\pi\text{ radians}}{\text{rev}} = 10\pi\ \frac{\text{radians}}{\text{min}} = \frac{d\theta}{dt}$

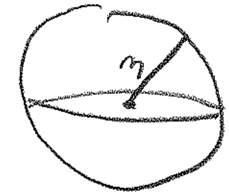
Now, $\frac{x}{\sqrt{3}} = \tan\theta \longrightarrow x = \sqrt{3}\tan\theta$

$$\Rightarrow \left.\frac{dx}{dt}\right|_{x=1} = \sqrt{3}\sec^2\theta\,\frac{d\theta}{dt}\Big| = (\sqrt{3})\left(\frac{2}{\sqrt{3}}\right)^2(10\pi) = \frac{40\sqrt{3}\,\pi}{3}$$

$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\boxed{\approx 40.41816646 \text{ KM/MIN}}$

(7a) $r = 3 \text{ cm} \pm .1 \text{ cm}$ for a sphere.
Estimate error w/ a differential in calculated volume. $V = \frac{4}{3}\pi r^3$



$$\Delta V \approx dV = 4\pi r^2 dr \longrightarrow$$

$$dV = 4\pi(3)^2(.1) = \boxed{3.6\pi \text{ cm}^3}$$

(7b) Relative Error $= \frac{\Delta V}{V} \approx \frac{dV}{V}$

$$\frac{4}{3}\pi(3)^2 = 12\pi = V \longrightarrow \frac{dV}{V} = \frac{3.6\pi}{12\pi} = .3$$

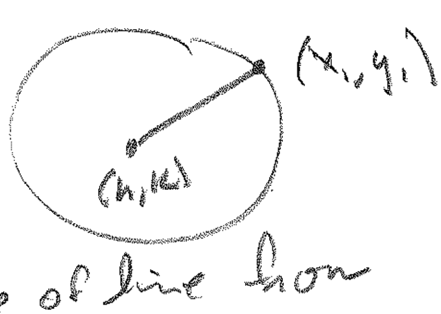
(7c) % Error $= \left(\frac{\Delta V}{V}\right)(100\%) \approx (.3)(100\%) = 30\%$

B1 Tan line @ P on circle is always $\perp$ to radius.

Pf

$$x^2 + y^2 = r^2$$
$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = -\frac{x}{y}$$

NOT CENTERED @ (h,k)! This is centered @ (0,0). WANT (h,k)



$\longrightarrow$ tan line @ (x_1, y_1) is

$$y = -\frac{x_1}{y_1}(x - x_1) + y_1, \text{ i.e.,}$$

$$\boxed{m_{tan} = -\frac{x_1}{y_1}}$$

Slope of line from (h,k) to (x_1, y_1) is

$\frac{y_1 - k}{x_1 - h}$ w/ $y = \frac{y_1 - k}{x_1 - h}(x_1 - h) + k$

201 TEST 2

B1 $(x-h)^2+(y-k)^2=r^2$

$2(x-h)+2(y-k)y'=0$

$2(y-k)y'=-2(x-h)$

$y'=-\frac{x-h}{y-k}$

@ (x_1,y_1), we have

$m_{tan}=-\frac{x_1-h}{y_1-k}=m_{tan}$

Since m from center to (x_1,y_1)

is $\frac{y_1-k}{x_1-h}=-\frac{1}{m_{tan}}$, we're done!

B2 Claim $\lim_{x\to 3}(x^2-2x+1)=4$

Scratch

Let $\varepsilon>0$. Assume $\delta\le 1$

Then $2<x<4$ if $0<|x-3|<\delta$

Now $|x^2-2x+1-4|=|x^2-2x-3|$

$=|x-3||x+1|<\delta|x+1|$

Need bound on $|x+1|$?

$2+1<x+1<4+1$

$3<x+1<5$, i.e.,

$|x+1|<5$ if $\delta\le 1$

$\boxed{\text{Pf}}$ Let $\varepsilon>0$

Define $\delta=\min\{1,\frac{\varepsilon}{5}\}$.

Then

$0<|x-3|<\delta$

$\Rightarrow |x^2-2x+1-4|$

$=|x+1||x-3|$

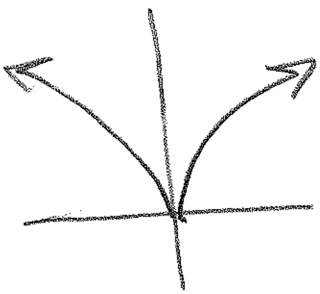
$<5|x-3|<5\delta$

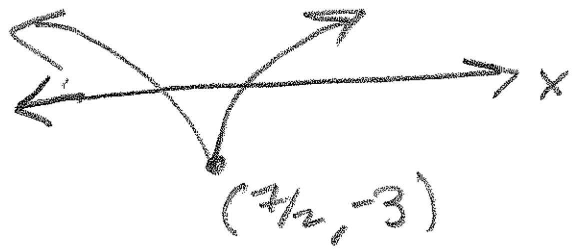
$\le 5\cdot\frac{\varepsilon}{5}=\varepsilon$ ■

201 TEST 2

B3 $y = 3(2x-7)^{2/3} - 3 = g(x)$

$x^{2/3} = f(x)$



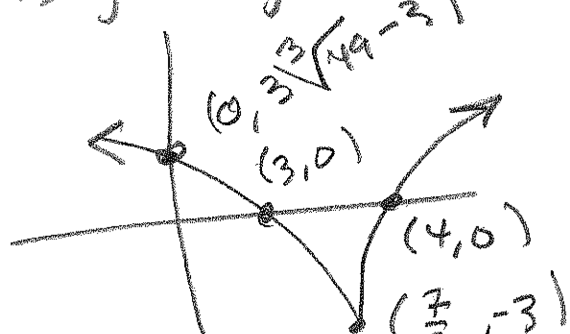


$g(0) = 3(-7)^{2/3} - 3$

$= 3\left[\sqrt[3]{49}\right] - 3$

> 0, since $\sqrt[3]{49} > 1$

This gives y-axis position:



x-int:

$(2x-7)^{2/3} = 1$

$\left((2x-7)^{\frac{1}{3}}\right)^2 = 1$

$(2x-7)^{\frac{1}{3}} = \pm 1$

$2x - 7 = \pm 1$

$2x = 7 \pm 1$

$x = \dfrac{7 \pm 1}{2} \rightarrow 4, \ 3$

B4 $f(x) = x^4 - 3x^3 - 22x^2 + 78x - 60$ has a zero in $(4,5)$, somewhere.

PG

$$\begin{array}{r|rrrrr} 4 & 1 & -3 & -22 & 78 & -60 \\ & & 4 & 4 & -72 & 24 \\ \hline & 1 & 1 & -18 & 6 & \boxed{-36 = f(4)} \end{array}$$

$$\begin{array}{r|rrrrr} 5 & 1 & -3 & -22 & 78 & -60 \\ & & 5 & 10 & -60 & 90 \\ \hline & 1 & 2 & -12 & 18 & \boxed{30 = f(5)} \end{array}$$

Polynomials (like f) are cnt^S, so

$f(4) = -36 < 0 < 30 = f(5) \longrightarrow$

$\exists\, c \in (4,5) \ni f(c) = 0$, by IVT.

B5 Approximate $\sin(48^\circ)$ with $L_1(x) =$ Tan line.

$f(x) = \sin x$, $x_1 = 45^\circ = \frac{\pi}{4}$ radians, $f(x_1) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Longrightarrow f'(x_1) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = m$

$L = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} \Longrightarrow \left\{ 48^\circ = 45^\circ + 3^\circ = \frac{\pi}{4} + \frac{3\pi}{180} = \frac{\pi}{4} + \frac{\pi}{60} \right.$

$L\left(48^\circ \cdot \frac{\pi}{180}\right) = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + \frac{\pi}{60} - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}$

$= \boxed{\frac{1}{\sqrt{2}}\left(\frac{\pi}{60}\right) + \frac{1}{\sqrt{2}}} \approx .7441308057$

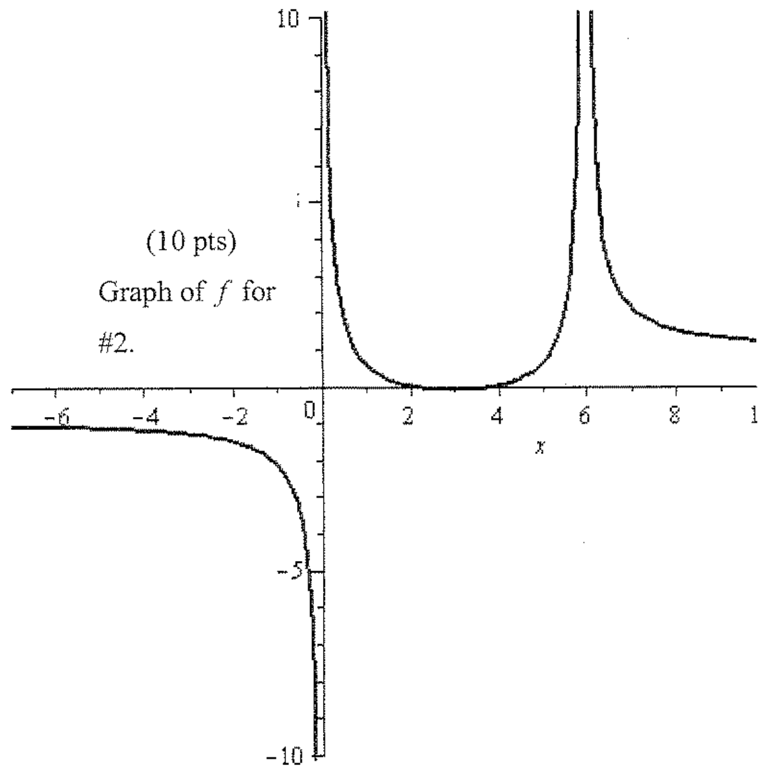
ACTUAL: $\approx .7431448255$

MAT 201 Test 2 – Spring, 2015 Name________________

100 Points Covers Chapter 2 NO GRAPHING CALCULATORS!!!

Show all work. Do your own work. Without supporting work, the slightest misstep leads to zero credit. Spread your work out! If you get stuck, start a fresh piece of paper. You can always *insert* more pages if you do it this way. No work should be on this cover sheet, except the graph for #2.

(10 pts) Graph of f for #2.



1. Let $f(x)=\sqrt{x+1}$.
   a. (5 pts) Find an equation of the tangent line to f at the point $(3, 2)$.
   b. (5 pts) Sketch a graph showing f and the tangent line to f at the point $(3, 2)$.
2. (10 pts) The graph of a function f is given on the right. On the same set of axes, sketch a graph of f'.
3. (5 pts each) Differentiate the following with respect to the independent variable.
   a. $f(x)=x^5-6x^{\frac{7}{3}}+6\sqrt[3]{x^7}+4x^{\frac{2}{5}}-\frac{3}{2}x^{-\frac{2}{3}}$
   b. $h(\omega)=(\omega^2+3\omega+13)(\omega^3-7\omega^2)$
   c. $H(t)=\dfrac{t^2+3t}{t^3+6t-11}$
   d. $g(x)=(x^2+3x+13)^3(x^3-7x^2)^{-5}$
   e. $r(x)=\dfrac{(x^2+3x)^3}{(x^3-7x^2)^5}$
   f. $Q(t)=\dfrac{\sin(t^2-3t)}{\cos(5t)}$
   g. $R(x)=\dfrac{\csc^3(5x)}{\tan(\pi x)}$
4. (10 pts) Show that $f(x)=x^3-6x^2+15x-7$ has no tangent line with a slope of $m=-2$.
5. Consider the relation $y\sin(2x)=x\cos(2y)$.
   a. (5 pts) Use implicit differentiation to find $y'=\dfrac{dy}{dx}$.
   b. (5 pts) Find an equation of the tangent line to the curve at the point $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

6. (10 pts) A lighthouse is located on a small island *exactly* $\sqrt{3}$ km from the nearest point P on a straight shoreline. The light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km away from P?

7. (10 pts) The radius of a sphere is 3 cm, with a possible error in measurement of 0.1 cm.
   a. Use differentials to estimate the error in the volume calculated from this measurement of the radius.

      $\left(\text{Hint: The volume of a sphere is given by } V=\frac{4}{3}\pi r^3\right)$.

   b. What is the relative error?
   c. What is the percent error?

Work up to 2 Bonus questions for up to 10 points extra.

Bonus (5 pts) Show, using implicit differentiation, that any tangent line to a circle, at a point $P=(x_1,y_1)$ on the circle, is perpendicular to the radius QP, where $Q=(h,k)$ is the center of the circle.

Bonus (5 pts) Prove that $\lim_{x\to 3}(x^2-2x+1)=4$.

Bonus (5 pts) Give a rough sketch of the graph of $y=3(2x-7)^{2/3}-3$, by transforming the graph of a basic function. Include x- and y-intercepts.

Bonus (5 pts) Convince me that $f(x)=x^4-3x^3-22x^2+78x-60$ has a zero in the interval $(4,5)$, without, you know, actually finding it.

Bonus (5 pts) Approximate $\sin(48^0)$ using the linearization.