

1. Arc Length and Area of Sector. Suppose we have a circle of radius $r = 7$.

a. (5 pts) Find the arc length on the circle, that is intercepted by an angle of 1220° . Round to 3 decimal places.

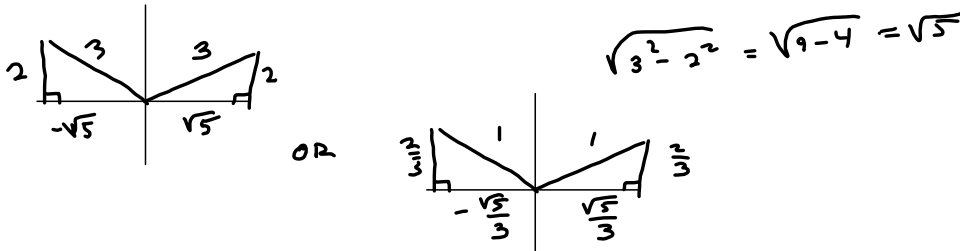
$$s = r\theta = 7(1220^\circ) \left(\frac{\pi}{180^\circ} \right) = \frac{427\pi}{9} \approx \boxed{149.0511}$$

b. (5 pts) Find the *exact* area of the sector that is intercepted (swept through) by an angle of $\theta = \frac{7\pi}{6}$

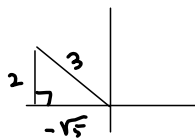
$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(7^2)\left(\frac{7\pi}{6}\right) = \boxed{\frac{343\pi}{12}} \approx 89.79719002$$

2. Answer the questions about the equation $\sin(\theta) = \frac{2}{3}$.

a. (5 points) Sketch two triangles that satisfy $\sin(\theta) = \frac{2}{3}$.



b. (5 pts) Assume the terminal side of the angle θ lies in the 2nd quadrant. Find the other five trigonometric functions of θ .



$$\sin \theta = \frac{2}{3}$$

$$\cos \theta = -\frac{\sqrt{5}}{3}$$

$$\tan \theta = -\frac{2}{\sqrt{5}} \text{ or } -\frac{2\sqrt{5}}{5}$$

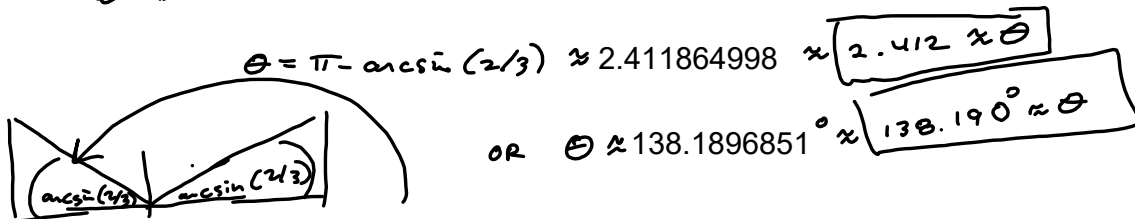
$$\csc \theta = \frac{3}{2}$$

$$\sec \theta = -\frac{3}{\sqrt{5}} \text{ or } -\frac{3\sqrt{5}}{5}$$

$$\cot \theta = -\frac{\sqrt{5}}{2}$$

c. (5 pts) Again, assuming θ 's terminal side lies in Q II, and $0 \leq \theta < 2\pi$, find θ , in radians *and* degrees, rounded to 3 decimal places.

$$\theta = \pi - \arcsin(2/3)$$



d. (5 pts) Give *all* solutions to the equation $\sin(\theta) = \frac{2}{3}$, in degrees *and* radians, rounded to three (3) decimal places.

$$\left\{ \arcsin(2/3) + 2\pi n \mid n \in \mathbb{Z} \right\} \approx \left\{ 0.7297276563 + 2\pi n \mid n \in \mathbb{Z} \right\}$$

$$\approx \left\{ .730 + 2\pi n \mid n \in \mathbb{Z} \right\} \cup \left\{ 2.412 + 2\pi n \mid n \in \mathbb{Z} \right\}$$

OR

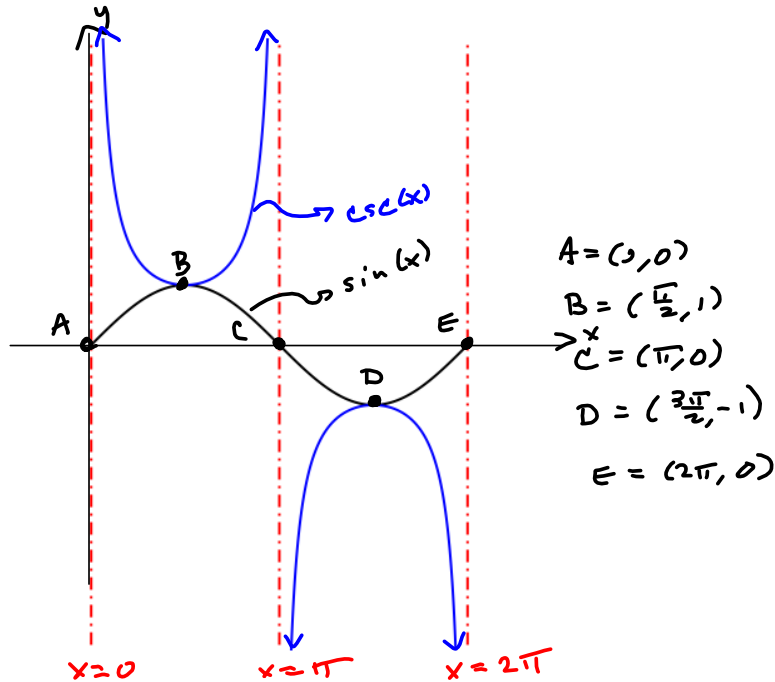
$$\left\{ 41.810^\circ + 360^\circ n \mid n \in \mathbb{Z} \right\} \cup \left\{ 138.190^\circ + 360^\circ n \mid n \in \mathbb{Z} \right\}$$

Other ways of writing it:

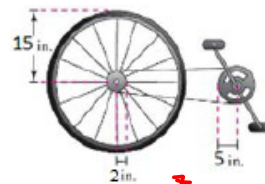
$$\left\{ x + 2\pi n \mid x = .730 \text{ or } 2.412, n \in \mathbb{Z} \right\}$$

$$\left\{ x + 360^\circ n \mid x = 41.810^\circ \text{ or } 138.190^\circ, n \in \mathbb{Z} \right\}$$

3. (5 pts) Sketch one period of the graphs of $y = \sin(x)$ and $y = \csc(x)$ on the same set of coordinate axes.



4. The radii of the pedal sprocket, the wheel sprocket, and the wheel of the bicycle in the figure are 5 inches, 2 inches and 15 inches, respectively. A cyclist is pedaling at a rate of 1.4 revolutions per second.
- a. (5 pts) Find the speed of the bicycle in feet per second.



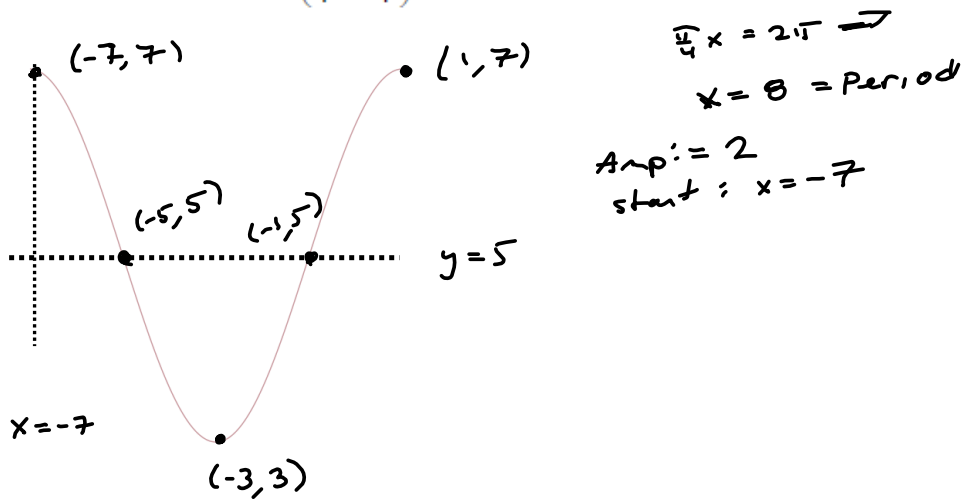
$$\left(\frac{1.4 \text{ rev pedal}}{\text{sec}} \right) \left(\frac{5 \text{ rev wheel}}{2 \text{ rev pedal}} \right) \left(\frac{2\pi}{1 \text{ rev}} \right) (15 \text{ in}) \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) = \left(\frac{7}{10} \right) \left(\frac{10\pi}{1} \right) \left(\frac{15}{12} \right)$$

$$= \frac{35\pi}{4} \frac{\text{ft}}{\text{s}} \approx 27.48893572 \frac{\text{ft}}{\text{s}} \approx \boxed{27.5 \frac{\text{ft}}{\text{s}}}$$

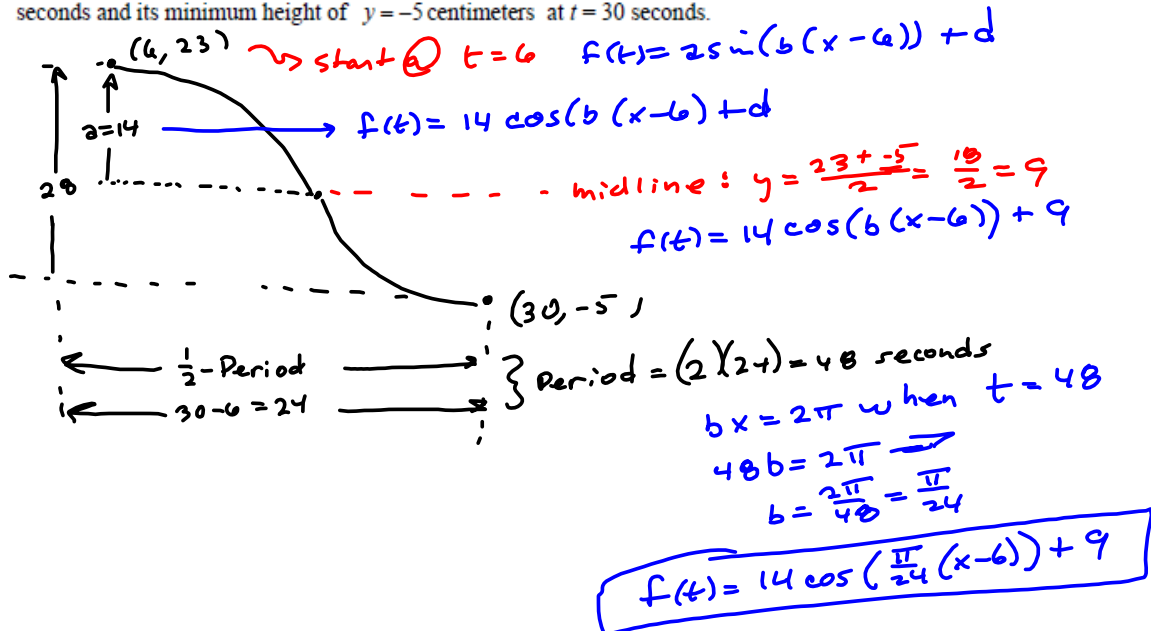
- b. (5 pts) Convert your answer, above, to miles per hour. Round final answers to 1 decimal place.

$$\left(\frac{35\pi}{4} \frac{\text{ft}}{\text{s}} \right) \left(\frac{60 \frac{\text{min}}{\text{hr}}}{5280 \frac{\text{ft}}{\text{mi}}} \right) = \frac{525\pi}{88} \frac{\text{mi}}{\text{hr}} \approx 18.74245617 \approx \boxed{18.7 \frac{\text{mi}}{\text{hr}}}$$

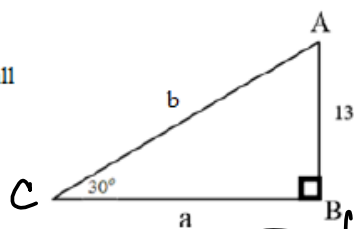
5. (5 pts) Sketch the graph of $f(x) = 2 \cos\left(\frac{\pi}{4}x + \frac{7\pi}{4}\right) + 5$.



6. (5 pts) Write the cosine function that achieves its maximum height of $y = 23$ centimeters at time $t = 6$ seconds and its minimum height of $y = -5$ centimeters at $t = 30$ seconds.



7. (5 pts) Solve the triangle in the figure on the right. That means, find all lengths and angles. Exact answers required.



$$\frac{13}{b} = \sin(30^\circ) = \frac{1}{2} \rightarrow$$

$$\boxed{2b = 13} \Rightarrow 2 = \sqrt{2b^2 - 13^2} = \sqrt{2^2 \cdot 13^2 - 13^2} = \sqrt{13^2(2^2 - 1)} = \boxed{13\sqrt{3} = 2}$$

$$\boxed{B = 90^\circ} \quad \boxed{A = 60^\circ}$$

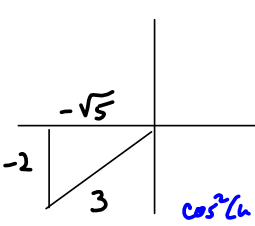
8. (5 pts) Find the exact value of $\arctan\left(\sin\left(\frac{3\pi}{2}\right)\right) = \arctan \theta$



$$= \arctan(-1) = \boxed{-\frac{\pi}{4}} = \boxed{-45^\circ}$$



9. (5 pts) Find $\sin\left(\frac{u}{2}\right)$, $\cos\left(\frac{u}{2}\right)$, and $\tan\left(\frac{u}{2}\right)$, given that $\sin(u) = -\frac{2}{3}$ and $\pi \leq u < \frac{3\pi}{2}$.

$\sin(u) = -\frac{2}{3}$, $u \in \text{QIII}$
 $\frac{\pi}{2} < \frac{u}{2} < \frac{3\pi}{4} \Rightarrow \frac{u}{2} \in \text{QII}$
 $\rightarrow \cos(u) = -\frac{\sqrt{5}}{3} \rightarrow (-, +)$
 $\sqrt{9-4} = \sqrt{5}$


Alternate:
 $\cos^2(u) = 1 - \sin^2(u) = 1 - \left(-\frac{2}{3}\right)^2$
 $= \frac{9-4}{9} = \frac{5}{9} \Rightarrow \cos(u) = -\frac{\sqrt{5}}{3}$ (QIII)

$\sin\left(\frac{u}{2}\right) = \frac{1 + \frac{\sqrt{5}}{3}}{2} = \sqrt{\frac{3+\sqrt{5}}{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$
 $= \frac{\sqrt{10+6\sqrt{5}}}{6} = \sin\left(\frac{u}{2}\right)$
 ≈ 0.9341723592

$\cos\left(\frac{u}{2}\right) = -\sqrt{\frac{1 - \frac{\sqrt{5}}{3}}{2}} = -\frac{\sqrt{10-6\sqrt{5}}}{6} = \cos\left(\frac{u}{2}\right)$
 ≈ -0.3568220899

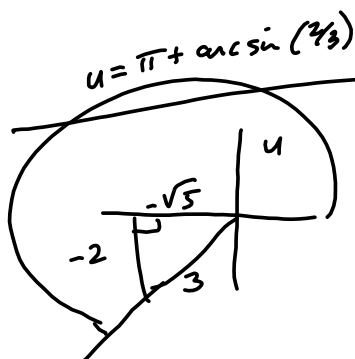
$\tan\left(\frac{u}{2}\right) = \frac{\sin u}{1 + \cos u} = \frac{-\frac{2}{3}}{1 - \frac{\sqrt{5}}{3}} = \frac{-2}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{-6-2\sqrt{5}}{4} = \frac{-3-\sqrt{5}}{2}$

OR

$\frac{1 - \cos(u)}{\sin(u)} = \frac{1 + \frac{\sqrt{5}}{3}}{-\frac{2}{3}} = \frac{3+\sqrt{5}}{-2} = \frac{-3-\sqrt{5}}{2}$

OR

$\tan\left(\frac{u}{2}\right) = \frac{\sin\left(\frac{u}{2}\right)}{\cos\left(\frac{u}{2}\right)} = \frac{\frac{\sqrt{10+6\sqrt{5}}}{6}}{-\frac{\sqrt{10-6\sqrt{5}}}{6}} = -\sqrt{\frac{10+6\sqrt{5}}{10-6\sqrt{5}}}$
 $= -\sqrt{\frac{(10+6\sqrt{5})^2}{(10-6\sqrt{5})(10+6\sqrt{5})}} = -\frac{10+6\sqrt{5}}{\sqrt{10^2-36 \cdot 5}} = -\frac{10+6\sqrt{5}}{\sqrt{6^2 \cdot 3^2 - 6^2 \cdot 5}}$
 $= -\frac{10+6\sqrt{5}}{6\sqrt{9-5}} = -\frac{3+\sqrt{5}}{\sqrt{4}} = -\frac{3+\sqrt{5}}{2}$
 $= \frac{-3-\sqrt{5}}{2} = \tan\left(\frac{u}{2}\right)$



$\frac{u}{2} = \frac{\pi + \arcsin\left(\frac{2}{3}\right)}{2}$

≈ -2.618033988

10. Consider the equation $4\sin^2(x) - 3 = 0$.

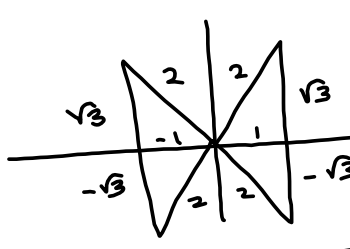
a. (5 pts) Find all solutions x , in radians *and* degrees, to the equation in the interval $[0, 2\pi)$.

b. (5 pts) Find all real solutions x , in radians *and* degrees.

$$\textcircled{a} \Rightarrow 4\sin^2(x) = 3$$

$$\Rightarrow \sin^2(x) = \frac{3}{4}$$

$$\Rightarrow \sin(x) = \pm \frac{\sqrt{3}}{2}$$



$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$x \in \{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \}$$

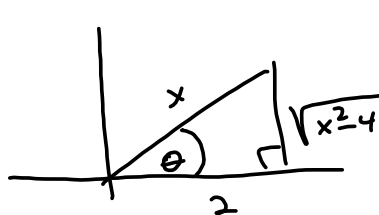
$$x \in \{ 60^\circ, 120^\circ, 240^\circ, 300^\circ \}$$

$$\textcircled{b} \quad x \in \{ x + n\pi \mid x \in \{ \frac{\pi}{3}, \frac{2\pi}{3} \}, n \in \mathbb{Z} \}$$

OR

$$x \in \{ x + 360^\circ n \mid x \in \{ 60^\circ, 120^\circ \}, n \in \mathbb{Z} \}$$

11. (5 pts) Draw the picture and use it to re-write $\sin\left(\cos^{-1}\left(\frac{2}{x}\right)\right)$ as an algebraic expression.



$$= \sin \theta$$

$$\Rightarrow \sin \theta = \frac{\sqrt{x^2 - 4}}{x}$$

12. Find the exact value of $\cos\left(\frac{5\pi}{12}\right)$ in two ways: (Hint: If degrees are easier for you, use degrees.)

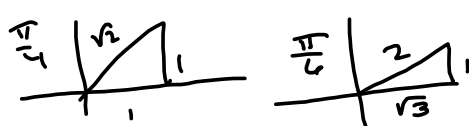
a. (5 pts) Use a Sum identity.

$$\textcircled{a} \quad 5 = 4 + 1 = 3 + 2 \checkmark$$

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \text{ OR } \frac{\sqrt{6} - \sqrt{2}}{4}$$



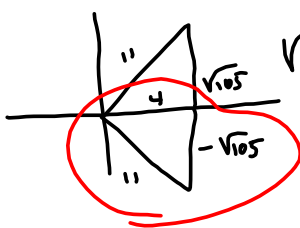
b. (5 pts) Use a Half-Angle identity

$$\frac{5\pi}{12} = \frac{1}{2} \cdot \frac{5\pi}{6} = \frac{1}{2}u, \text{ where } u = \frac{5\pi}{6}$$

$$\cos\left(\frac{u}{2}\right) = \sqrt{\frac{1 + \cos(u)}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2} = \cos\left(\frac{5\pi}{12}\right)$$



13. (5 pts) Find $\sin(2u)$, $\cos(2u)$ and $\tan(2u)$, given that $\cos(u) = \frac{4}{11}$ and $\sin(u) < 0$.



$$\sqrt{11^2 - 4^2} = \sqrt{121 - 16} = \sqrt{105}$$

$$\sin(2u) = 2\sin u \cos u = 2\left(-\frac{\sqrt{105}}{11}\right)\left(\frac{4}{11}\right) = \boxed{-\frac{8\sqrt{105}}{121} = \sin(2u)}$$

-0.6774843484

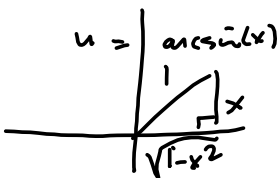
$$\cos(2u) = 2\cos^2(u) - 1 = 2\left(\frac{4}{11}\right)^2 - 1 = \frac{32}{121} - \frac{121}{121} = \boxed{-\frac{89}{121} = \cos(2u)}$$

-0.7355371903

$$\tan(2u) = \frac{-\frac{8\sqrt{105}}{121}}{-\frac{89}{121}} = \boxed{\frac{8\sqrt{105}}{89} = \tan(2u)}$$

0.9210742255

14. BONUS (5 pts) Re-write $\cos(\arcsin(x) + \arccos(x))$ as an algebraic expression. (Hint: Use Sum identity.)



$$\begin{aligned} &= \cos(u + v) = \cos u \cos v - \sin u \sin v \\ & \quad u = \arcsin(x) \quad v = \arccos(x) \\ &= (\sqrt{1-x^2})(x) - x\sqrt{1-x^2} \\ &= \boxed{0} \end{aligned}$$
