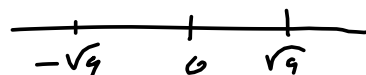


4.1 - Complex Numbers

Real Number Line and Real Numbers = \mathbb{R} Principle Square Root \sqrt{x} is the positive # satisfying $y^2 = x$ $y^2 = x$ has 2 solutions $y = \pm\sqrt{x}$ Imaginary Unit is $i = \sqrt{-1}$ Notice

$$i^2 = (\sqrt{-1})^2 = -1$$



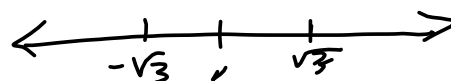
$$(-\sqrt{9})^2 = 9$$

$$(\sqrt{9})^2 = 9$$

Pure Imaginary Number -

$$2i = 2\sqrt{-1} = \sqrt{-4}$$

$$= \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = (i)(2) = 2i$$



$$(-\sqrt{3})^2 = \sqrt{3}^2 = 3$$

Complex Number in Standard Form.

$$z = a + bi, \text{ where } a, b \in \mathbb{R}$$

$$3 + 2i, 5 - 7i, -6 + 2i, -3 - 2i$$

$$a = \operatorname{Re}(z) = \text{real part of } z$$

$$b = \operatorname{Im}(z) = \text{imaginary part}$$

Conjugate of a Complex Number z .

~

$$z = a + bi, \text{ then } \bar{z} = a - bi$$

$$z = 3 + 2i \rightarrow \bar{z} = 3 - 2i$$

Multiplication of Complex Numbers ("FOIL" them.)

$$\boxed{(3+2i)(3-2i)} = \text{Distributive Law!}$$

$$= 3(3) - (3)(2i) + (2i)(3) + (2i)(2i)$$

$$= 9 - 6i + 6i - 4i^2$$

$$= 9 - \cancel{6i} + \cancel{6i} - 4(-1) = 9 + 4 = 13 \quad ! = |z|^2$$

$$\boxed{(x+3)(x^2-2x-1)}$$

$$(2-3i)(5+2i) = 10 - 11i - 6i^2$$

$$= 10 - 11i + 6$$

$$= \boxed{16 - 11i}$$

Magnitude of a Complex Number. $|z| = \sqrt{a^2+b^2} = |a+bi|$

Recall $\vec{v} = \langle a, b \rangle \Rightarrow \|\vec{v}\| = \sqrt{a^2+b^2} = \sqrt{\vec{v} \cdot \vec{v}}$

$$|z| = \sqrt{z \bar{z}}$$

Writing a Complex Number in Standard Form - "Conjugate Trick"

$$z = 2+3i, \quad w = 9-2i \Rightarrow$$

$$z + w = 11 + i$$

$$3+2i + 6i + 4i^2$$

$$\left(\frac{1+2i}{3-2i} \right) = \left(\frac{1+2i}{3-2i} \right) \left(\frac{3+2i}{3+2i} \right) = \frac{1+8i}{13} = \frac{1}{13} + \frac{8}{13}i$$

4.2 - Recall from College Algebra:

Factor Theorem

Factor Theorem - If $x-c$ is a factor of a polynomial, then $x=c$ is a zero.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad \& \quad P(c) = 0 \iff$$

$$P(x) = (x-c)(b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0)$$

You just split off a factor of $x-c$

The way we do it is by Synthetic Division.

$P(x) = x^4 - 3x^3 + 3x^2 + 37x - 78$ has a zero $\textcircled{2}$ $x=2$.

We split off a factor of $x-2$: Use Division:

$$\begin{array}{r}
 x^3 - x^2 + x + 39 \quad r \ 0 = P(2)! \\
 x-2 \overline{) x^4 - 3x^3 + 3x^2 + 37x - 78} \\
 \underline{-(x^4 - 2x^3)} \\
 -x^3 + 3x^2 + 37x - 78 \\
 \underline{-(-x^3 + 2x^2)} \\
 x^2 + 37x - 78 \\
 \underline{-(x^2 - 2x)} \\
 39x - 78 \\
 \underline{-(39x - 78)} \\
 0
 \end{array}
 \quad
 \begin{array}{l}
 \frac{x^4}{x} = x^3 \\
 -\frac{x^3}{x} = -x^2 \\
 \frac{x^2}{x} = x \\
 \frac{39x}{x} = 39
 \end{array}$$

This says $P(x) = \overset{0}{(x-2)(x^3 - x^2 + x + 39)} + 0$
↑ Divisor ↓ Quotient = "Depressed Polynomial!"

Synthetically:

$$\begin{array}{r|rrrrrr}
 2 & 1 & -3 & 3 & 37 & -78 \\
 & & 2 & -2 & 2 & 78 \\
 \hline
 -3 & 1 & -1 & 1 & 39 & 0 = P(2) \text{ Sweet!} \\
 & & -3 & 12 & -39 & \\
 \hline
 & 1 & -4 & 13 & 0 = P(-3) \text{ Sweet!} &
 \end{array}$$

$$\text{So } P(x) = (x-2)(x+3)(x^2-4x+13)$$

The $x^2-4x+13$ is irreducible over the reals.

$$b^2-4ac = 4^2-4(1)(13) = 16-52 = -36 < 0 \rightarrow \text{NO Real Zeros!}$$

$$a=1, b=-4, c=13$$

$$i\sqrt{36} = i \cdot 6 = 6i$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm 6i}{2(1)} = \frac{2 \pm 3i}{1} = 2 \pm 3i \rightarrow$$

$P(x) = (x-2)(x+3)(x-(2+3i))(x-(2-3i))$ has been split into linear factors!

Complete the square:

$$\begin{aligned}
 x^2-4x+13 &= x^2-4x+2^2-4+13 \\
 &= (x-2)^2+9 \stackrel{\text{SET}}{=} 0
 \end{aligned}$$

$$\Rightarrow (x-2)^2 = -9$$

$$x-2 = \pm\sqrt{-9} = \pm 3i$$

$$\rightarrow \boxed{x = 2 \pm 3i}$$

Conjugate-Pairs Theorem - $P(x)$ has real coefficients and $a+bi$ is a zero
 $\Rightarrow (a-bi)$ is a zero

Fundamental Theorem of Algebra - Every Polynomial of degree $n \geq 1$ has a zero (complex)

Linear Factorization Theorem. - Apply FTA n times \rightarrow
 we can split ANY polynomial into linear factors:

$$P(x) = a_n (x-r_1)(x-r_2)\dots(x-r_n)$$

Repeated-Roots Version:

$$P(x) = a_n (x-r_1)^{m_1} (x-r_2)^{m_2} \dots (x-r_k)^{m_k}$$

where $m_1 + m_2 + \dots + m_k = n$

$$x^9 - 12x^8 - 6x^7 + 540x^6 - 1239x^5 - 7464x^4 + 27780x^3 + 20400x^2 - 168000x + 160000$$

$$= (x-2)^2 (x+4)^3 (x-5)^4$$

Factor into the product of linear factors:

$$2x^3 - x^2 + 96x - 48$$

Old-School

$$\pm 1, \pm \frac{1}{2}, \pm 2, \pm \frac{3}{2}, \pm 3, \pm \frac{3}{2}$$

Turns out $x = \frac{1}{2}$ is a zero

(Use grapher).

$$\begin{array}{r} 2 \overline{) 48} \\ \underline{24} \\ 24 \\ \underline{12} \\ 12 \\ \underline{6} \\ 6 \\ \underline{3} \\ 0 \end{array} \quad 2^4 \cdot 3 = 48$$

Given $1+i$ is a zero of $x^4 - 10x^2 + 34x - 48x + 32 = P(x)$

Find all zeros of $P(x)$,

2 methods:

My way Split off factor of $(x - (1+i))$ & $(x - (1-i))$

$$\begin{array}{r} 1+i \overline{) 1 \quad -10 \quad 34 \quad -48 \quad 32} \\ \hline \end{array}$$

$$\begin{array}{r} 1-i \overline{) \\ \hline 0 \end{array}$$

$$\begin{array}{cccc} & a & b & c & 0 \end{array}$$

Leaving $ax^2 + bx + c = 0$ to find last 2 roots.

Book way:

$$(x - (1+i))(x - (1-i)) = (x - 1 - i)(x - 1 + i)$$

$$= x^2 - x + \underline{i}x - x + 1 - \underline{i} \quad \underline{-ix} + \underline{i} - i^2$$

$$= x^2 - 2x - i^2 = x^2 - 2x + 2$$

$$x^2 - 2x + 2 \overline{) x^4 - 10x^2 + 34x - 48x + 32}$$

Quotient is a quadratic

Y.1 Dur Friday, Y.2 Dur Monday

Given $1+i$ is a zero, $1-i$ is a zero, by Conjugate Pairs Theorem

$$\begin{array}{r}
 x^2 - 8x + 16 \quad | \quad 0 \\
 x^2 - 2x + 2 \overline{) x^4 - 10x^3 + 34x^2 - 48x + 32} \\
 \underline{-(x^4 - 2x^3 + 2x^2)} \\
 \phantom{x^2 - 2x + 2 \overline{) }} - 8x^3 + 32x^2 - 48x + 32 \\
 \underline{-(-8x^3 + 16x^2 - 16x)} \\
 \phantom{x^2 - 2x + 2 \overline{) }} 16x^2 - 32x + 32 \\
 \underline{-(16x^2 - 32x + 32)} \\
 \phantom{x^2 - 2x + 2 \overline{) }} 0
 \end{array}$$

$$\frac{x^4}{x^2} = x^2$$

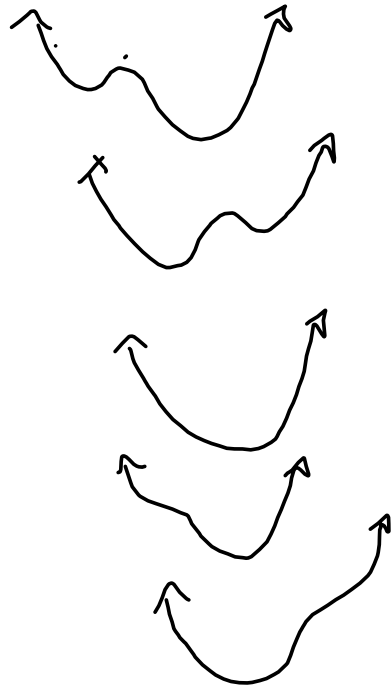
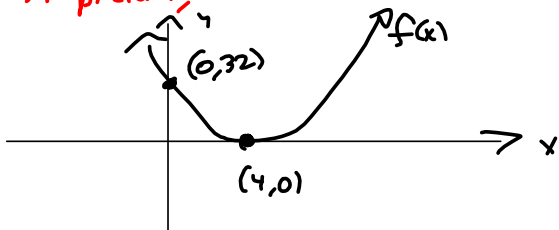
$$\begin{aligned}
 (a+b)^2 &= (a^2 + 2ab + b^2) \\
 (a-b)^2 &= (a^2 - 2ab + b^2)
 \end{aligned}$$

This says $f(x) = (x^2 - 2x + 2)(x^2 - 8x + 16)$
 $= (x^2 - 2x + 2)(x - 4)(x + 4)$

$= (x - (1+i))(x - (1-i))(x - 4)(x + 4) = f(x)$, split into linear factors.

Zeros are $1+i, 1-i, 4$
 multiplicity of $x=4$ is 2 .
 That's from $(x-4)^2$.

A picture, based on this info: $2x^4 + \dots$



$$x^2 - 2x + 2 \sqrt{x^4 - 10x^2 + 34x - 48x + 32}$$

Use $x = 1+i$ is a zero

$$\begin{array}{r}
 \underline{1+i} \mid 1 \quad -10 \quad 34 \quad -48 \quad 32 \\
 \quad \quad \quad 1+i \quad -10-8i \quad 32+16i \quad -32 \\
 \hline
 \underline{1-i} \mid 1 \quad -9+i \quad 24-8i \quad -16+16i \quad 0 \text{ sweet!} \\
 \quad \quad \quad 1-i \quad -8+8i \quad 16-16i \\
 \hline
 1 \quad -8 \quad 16 \quad 0 \text{ sweet!}
 \end{array}$$

This shows $f(x) = (x - (1+i))(x - (1-i))(x^2 - 8x + 16)$

$$(1+i)(24-8i) = 24 - 8i + 24i - 8i^2$$

$$= 24 + 16i + 8$$

$$= 32 + 16i$$

$$(1+i)(-16+16i)$$

$$= -16(1+i)(1-i) = -16(i^2 + 1) = -32$$

$$(2+bi)(2-bi) = a^2 + b^2 = |a+bi|^2$$

(Recall: $\|\bar{u}\|^2 = \bar{u} \cdot u = a^2 + b^2$, when $\bar{u} = \langle a, b \rangle$)

$$(1+i)(-9+i)$$

$$= -9 + i - 9i + i^2$$

$$= -9 - 8i - 1$$

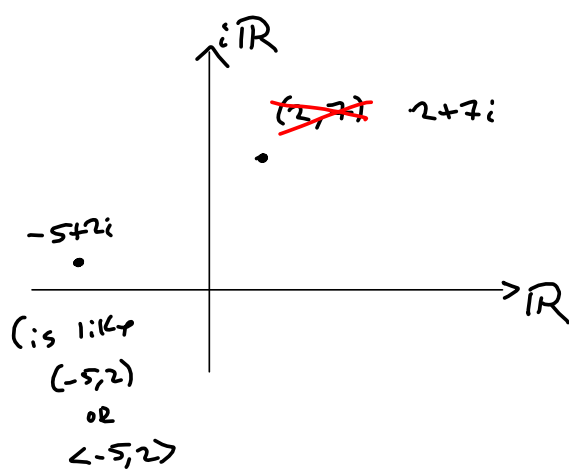
Find a polynomial function with the given zeros:
3, 2, $1+i$
with real coefficients

Written Test - $(x-3)(x-2)(x-(1+i))(x-(1-i))$

Wub/Assign. - *wants it multiplied out*

$$x^4 - 7x^3 + 18x^2 - 22x + 12$$

The Complex Plane.



Write $\frac{1-3i}{4+2i}$ in standard form: $(a+bi)$

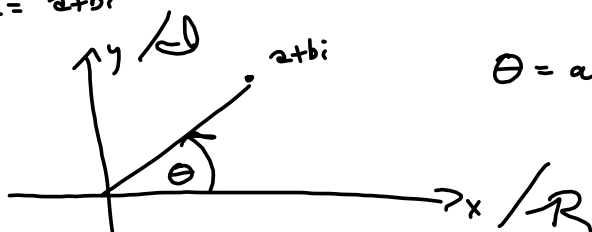
Conjugate Trick:

$$\left(\frac{1-3i}{4+2i}\right)\left(\frac{4-2i}{4-2i}\right) = \frac{4-14i-6}{4^2+2^2} = \frac{-2-14i}{20} = -\frac{2}{20} - \frac{14}{20}i$$

$$= \boxed{-\frac{1}{10} - \frac{7}{10}i}$$

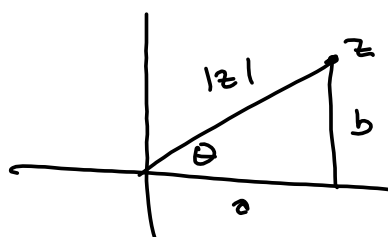
Trigonometric form of a complex # $z = a+bi$:

$$z = a+bi$$



$\theta =$ argument of z
is like the direction
angle for vectors!
Thanks, Alex B.

$$z = |z| (\cos \theta + i \sin \theta)$$



This is analogous to

$$a = |z| \cos \theta$$

$$b = |z| \sin \theta$$

$$\vec{u} = \|\vec{u}\| \langle \cos \theta, \sin \theta \rangle$$

Build an example

$$z = 3 \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right), |z| = 3$$

$$\sqrt{2} \sqrt{3} = 3 \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right)$$

$$= \boxed{\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i} = z$$

$$w = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)$$

$$= 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) = \boxed{1 + \sqrt{3} i} = w, |w| = 2$$

Multiply them together

$$\left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i \right) (1 + \sqrt{3} i)$$

$$= \frac{3}{2} (\sqrt{2} + \sqrt{2} i) (1 + \sqrt{3} i)$$

$$= \frac{3}{2} (\sqrt{2} + \sqrt{6} i + \sqrt{2} i + \sqrt{6} i^2)$$

$$= \frac{3}{2} ((\sqrt{2} - \sqrt{6}) + (\sqrt{6} + \sqrt{2}) i)$$

$$\frac{3\sqrt{2}}{2} + \frac{3i\sqrt{2}\sqrt{3}}{2} + \frac{3i\sqrt{2}}{2} - \frac{3\sqrt{2}\sqrt{3}}{2}$$

$$\approx -1.552914271 + 5.795554957i = zw$$

$$|zw| \approx 5.999999999 \approx 6 = |z||w|$$

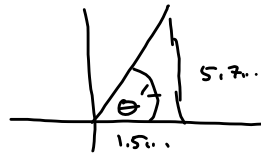
The magnitude of the product is the product of the respective magnitudes.

What's the argument (direction angle) of zw ?

$$-1.5529 + 5.7956i$$



$$\theta = \pi - \arctan(5.7... / 1.5...)$$



$$\pi - \arctan\left(\frac{5.795554957}{1.552914271}\right)$$

$$1.832595715$$

$$\frac{\% \cdot 180}{\pi}$$

$$105.0000000^\circ = 45^\circ + 60^\circ$$

$$z = 3 \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)$$

$$w = 2 \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3} \right)$$

$$\rightarrow zw = 6 \left(\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \right)$$

$$z^{10} = 3^{10} \left(\cos\left(\frac{10\pi}{4}\right) + i \sin\left(\frac{10\pi}{4}\right) \right)$$