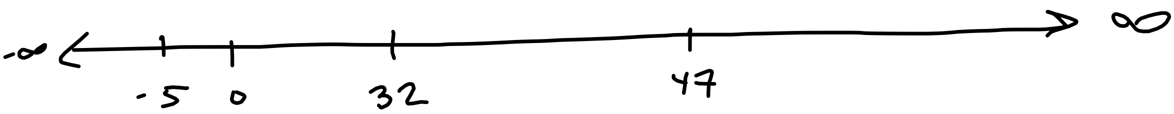
4.1 - Complex Numbers

Real Number Line and Real Numbers $= \mathbb{R}$

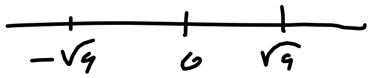


Principle Square Root $\sqrt{x}$ is the positive # satisfying $y^2 = x$

$y^2 = x$ has 2 solutions $y = \pm\sqrt{x}$

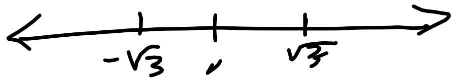
Imaginary Unit is $i = \sqrt{-1}$ Notice

$i^2 = (-i)^2 = -1$



$(-\sqrt{9})^2 = 9$

$(\sqrt{9})^2 = 9$



$(-\sqrt{3})^2 = \sqrt{3}^2 = 3$

Pure Imaginary Number _

$2i = 2\sqrt{-1} = \sqrt{-4}$

$= \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = (i)(2) = 2i$

Complex Number in Standard Form.

$z = a+bi$, where $a, b \in \mathbb{R}$

$3+2i$, $5-7i$, $-6+2i$, $-3-2i$

$a = \mathrm{Re}(z)$ = real part of z

$b = \mathrm{Im}(z)$ = imaginary part

Conjugate of a Complex Number z. ~

$z = a+bi$, then $\bar{z} = a-bi$

$z = 3+2i \longrightarrow \bar{z} = 3-2i$

Multiplication of Complex Numbers ("FOIL" them.)

$(3+2i)(3-2i) =$ Distributive Law!

$= 3(3) - (3)(2i) + (2i)(3) - (2i)(2i)$

$= 9 - 6i + 6i - 4i^2$

$= 9 - 4(-1) = 9+4 = 13$! $= |z|^2$

$(x+3)(x^2 - 2x - 1)$

$(2-3i)(5+2i) = 10 - 11i - 6i^2$

$= 10 - 11i + 6$

$= \boxed{16 - 11i}$

Magnitude of a Complex Number. $|z| = \sqrt{a^2+b^2} = |a+bi|$

Recall $\vec{v} = \langle a, b \rangle \Longrightarrow \|\vec{v}\| = \sqrt{a^2+b^2} = \sqrt{\vec{v} \bullet \vec{v}}$

$|z| = \sqrt{z\bar{z}}$

Writing a Complex Number in Standard Form - "Conjugate Trick"

$z = 2+3i,\ W = 9-2i \Longrightarrow$

$Z + W = 11 + i$

$3+2i + 6i + 4i^2$

$\left(\frac{1+2i}{3-2i}\right) = \left(\frac{1+2i}{3-2i}\right)\left(\frac{3+2i}{3+2i}\right) = \frac{-1+8i}{13} = -\frac{1}{13} + \frac{8}{13}i$

4.2 - Recall from College Algebra:

Factor Theorem

FACTOR Theorem - If $x-c$ is a factor of a polynomial, then $x=c$ is a zero.

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ & $P(c)=0 \longleftrightarrow$

$P(x) = (x-c)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1 x + b_0)$

You just split off a factor of $x - c$

The way we do it is by Synthetic Division.

$P(x) = x^4 - 3x^3 + 3x^2 + 37x - 78$ has a zero @ $x=2$.

We split off a factor of $x-2$: Use Division:

$$
\begin{array}{r}
x^3 - x^2 + x + 39 \quad r\ 0 = P(2)! \\
x-2 \,\big)\, x^4 - 3x^3 + 3x^2 + 37x - 78 \\
-(x^4 - 2x^3) \qquad\qquad\qquad\qquad \\
\hline
-x^3 + 3x^2 + 37x - 78 \\
-(-x^3 + 2x^2) \qquad\qquad \\
\hline
x^2 + 37x - 78 \\
-(x^2 - 2x) \qquad \\
\hline
39x - 78 \\
-(39x - 78) \\
\hline
0
\end{array}
$$

$\frac{x^4}{x} = x^3$

$-\frac{x^3}{x} = -x^2$

$\frac{x^2}{x} = x$

$\frac{39x}{x} = 39$

This says $P(x) = \underset{\text{Divisor}}{(x-2)}\underbrace{(x^3 - x^2 + x + 39)}_{\text{Quotient = "Depressed Polynomial"}} + 0$

Synthetically:

$$\begin{array}{r|rrrrr} 2 & 1 & -3 & 3 & 37 & -78 \\ & & 2 & -2 & 2 & 78 \\ \hline -3 & 1 & -1 & 1 & 39 & \boxed{0 = P(2)} \\ & & -3 & 12 & -39 & \\ \hline & 1 & -4 & 13 & \boxed{0 = P(-3)} & \end{array}$$

Sweet! Sweet!

So $P(x) = (x-2)(x+3)(x^2-4x+13)$

The $x^2-4x+13$ is irreducible over the reals.

$b^2-4ac = 4^2-4(1)(13) = 16-52 = -36 < 0 \rightarrow$ No Real Zeros!

$a=1,\ b=-4,\ c=13$ $\qquad i\sqrt{36} = i\cdot 6 = 6i$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{4 \pm 6i}{2(1)} = \dfrac{2 \pm 3i}{1} = 2 \pm 3i \longrightarrow$

$P(x) = (x-2)(x+3)(x-(2+3i))(x-(2-3i))$ has been split into linear factors!

Complete the square:

$x^2-4x+13 = x^2-4x+2^2-4+13$

$= (x-2)^2+9 \overset{\text{SET}}{=} 0$

$\Rightarrow (x-2)^2 = -9$

$x-2 = \pm\sqrt{-9} = \pm 3i$

$\rightarrow \boxed{x = 2 \pm 3i}$

Conjugate-Pairs Theorem – $P(x)$ has real coefficients and $a+bi$ is a zero
$\Rightarrow$ $(a-bi)$ is a zero

Fundamental Theorem of Algebra – Every Polynomial of degree $n \geq 1$ has a zero (complex)

Linear Factorization Theorem. – Apply FTA n times $\longrightarrow$
we can split ANY polynomial into linear factors:

$$P(x) = a_n (x-r_1)(x-r_2)\cdots(x-r_n)$$

Repeated-Roots Version:

$$P(x) = a_n (x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}$$

where $m_1+m_2+\dots+m_k = n$

$$x^9 - 12x^8 - 6x^7 + 540x^6 - 1239x^5 - 7464x^4 + 27780x^3 + 20400x^2 - 168000x + 160000$$

$$= (x-2)^2(x+4)^3(x-5)^4$$

Factor into the product of linear factors:

$2x^3 - x^2 + 96x - 48$

Old-School

$\pm 1, \pm\frac{1}{2}, \pm 2, \pm\frac{2}{2}, \pm 3, \pm\frac{3}{2}$

$$\begin{array}{r|l} 2 & 48 \\ 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ & 3 \end{array} \qquad 2^4 \cdot 3 = 48$$

Turns out $x = \frac{1}{2}$ is a zero

(Use grapher).

Given $1+i$ is a zero of $x^4 - 10x^3 + 34x^2 - 48x + 32 = P(x)$

Find all zeros of $P(x)$.

2 methods:

My WAY split off factor of $(x-(1+i))$ & $(x-(1-i))$

$$\begin{array}{r|rrrrr} 1+i & 1 & -10 & 34 & -48 & 32 \\ \hline 1-i & & & & & 0 \\ \hline & & a & b & c & 0 \end{array}$$

Leaving $ax^2+bx+c=0$ to find last 2 roots.

Book way:

$(x-(1+i))(x-(1-i)) = (x-1-i)(x-1+i)$

$= x^2 - x + ix - x + 1 - i - ix + i - i^2$

$= x^2 - 2x - i^2 = x^2 - 2x + 1$

$$x^2-2x+1 \overline{\big)\, 2x^3 - x^2 + 96x - 48}$$

Quotient is a quadratic

4.1 Due Friday, 4.2 Due Monday

Find a polynomial Function with the given zeros:
3, 2, 1+i

Written Test

WebAssign.