Complex #s Review

$z = a+bi$ and $w = c+di$

$= \langle a,b \rangle = (a,b) \qquad \langle c,d \rangle = (c,d)$

Then $z+w = (a+b) + (c+d)i$

$zw = (a+bi)(c+di) = ac + adi + bci + bd\,i^2$

$= (ac-bd) + (ad+bc)i$

$z = 3+2i,\ w = 5-7i \longrightarrow$

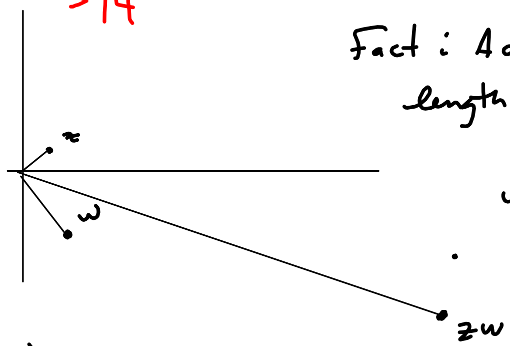
zw ← $z+w$ = $15-21i+10i-21i^2$ → $-14\,i^2$

dummy

$= (15+21) + (-21+10)i = 36 - 11i$

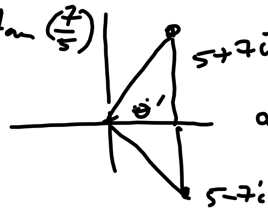
14   29

Fact: Add angles & multiply lengths!



$\theta_z = \arctan\left(\frac{2}{3}\right)$

$\theta_w = -\arctan\left(\frac{7}{5}\right)$ OR $2\pi - \arctan\left(\frac{7}{5}\right)$



$\arctan\left(\frac{7}{5}\right) = \theta'$

$\theta_{zw} = \theta_z + \theta_w$

$|zw| = |z||w|$

$z = |z|(\cos\theta_z + i\sin\theta_z)$

$w = |w|(\cos\theta_w + i\sin\theta_w)$

$zw = |z||w|\left(\cos(\theta_w+\theta_z) + i\sin(\theta_w+\theta_z)\right)$

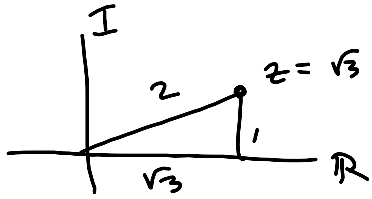
FOR A PURE ROTATION of z through an angle γ.

$(z)(\cos\gamma + i\sin\gamma) = (z)(e^{i\gamma})$

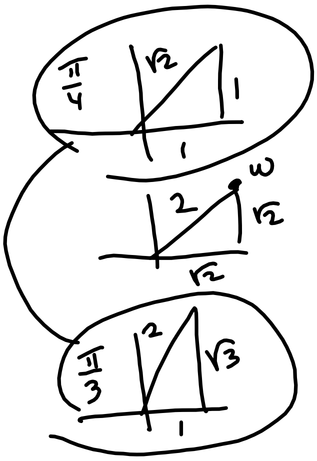
$-\arctan\left(\frac{7}{5}\right) + \arctan\left(\frac{2}{3}\right)$

Click here for Maple (Computer Algebra) for this.

Trigonometric Form



$z = \sqrt{3} + 1i \quad = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$



$w = \sqrt{2} + \sqrt{2}i = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$zw = 2\cdot 2\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)$

$= 4\left(\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\right.$

$\left. + i\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right)\right)$

$= 4\left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + i\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)\right)$

$= 4\left(\frac{1-\sqrt{3}}{2\sqrt{2}} + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right)$

$= 4\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\right) = \sqrt{2}-\sqrt{6} + (\sqrt{6}+\sqrt{2})i$

The old way $\quad (\sqrt{2}+\sqrt{2}i)(\sqrt{3}+1i)$

$= \sqrt{6} + \sqrt{2}i + \sqrt{6}i + \sqrt{2}i^2$

$= \sqrt{6} - \sqrt{2} + i(\sqrt{2}+\sqrt{6})$

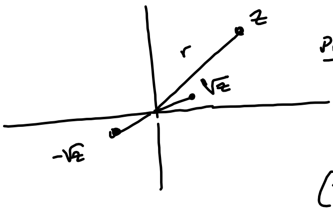
I dropped a sign, somewhere!

I'll try to find my mistake, but I'm just not seeing it and none of you are saving me.

Powers: If $z = r(\cos\theta + i\sin\theta)$, then

$$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$$
$$z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$$
$$\vdots$$
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Next up: n^{th} roots.



$z = r(\cos\theta + i\sin\theta)$

Principle n^{th} root of z is

$$\sqrt[n]{r}\left(\cos\left(\tfrac{\theta}{n}\right) + i\sin\left(\tfrac{\theta}{n}\right)\right)$$

$$\sqrt{z} = \sqrt{r}\left(\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)\right)$$

$$(\sqrt{z})^2 = z = r(\cos\theta + i\sin\theta) \checkmark$$

Note that $-\sqrt{z}$ is ALSO a square root of z

$$x^2 = 4 \Longrightarrow x = \pm\sqrt{4} = \pm 2$$

$$-\sqrt{z} = \sqrt{r}\left(\cos\left(\tfrac{\theta}{2} + \pi\right) + i\sin\left(\tfrac{\theta}{2} + \pi\right)\right)$$

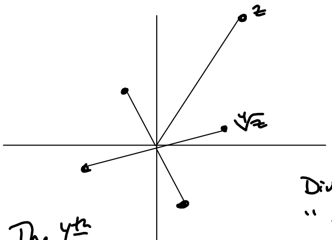
Square Roots of z:

$$\sqrt{z} = \sqrt{r}\left(\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)\right),$$
$$\sqrt{r}\left(\cos\left(\tfrac{\theta+2\pi}{2}\right) + i\sin\left(\tfrac{\theta+2\pi}{2}\right)\right)$$

There will be 3 cube roots, 4 4th roots, 5 5th roots, ...

4^{th} roots of z

$$\sqrt[4]{z} = \sqrt[4]{r}\left(\cos\left(\tfrac{\theta}{4}\right) + i\sin\left(\tfrac{\theta}{4}\right)\right)$$



Divide 2π by 4. That's the "increment."

The 4^{th} roots of z arranged around the circle.

$$z = 16\left(\cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right)\right)$$
$$\sqrt[4]{z} = 2\left(\cos\left(\tfrac{\pi}{12}\right) + i\sin\left(\tfrac{\pi}{12}\right)\right)$$

increment

$\frac{\pi}{3} + \frac{2\pi}{4} = \frac{4\pi}{12} + \frac{6\pi}{12} = \frac{10\pi}{12}$ 2

$\frac{10+6}{12} = \ldots \frac{16\pi}{12}$ 3

$\frac{\pi}{2}$, dummy

$\frac{22\pi}{12}$ 4

$$\frac{\pi}{3} + \frac{\pi}{2} = \frac{2\pi + 3\pi}{6} = \frac{5\pi}{6}$$
$$\frac{5\pi}{6} + \frac{3\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$$
$$\frac{8\pi}{6} + \frac{3\pi}{6} = \frac{11\pi}{6}$$

4^{th} roots of $z = 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

are

$$2\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right)$$
$$2\left(\cos\left(\tfrac{5\pi}{6}\right) + i\sin\left(\tfrac{5\pi}{6}\right)\right)$$
$$2\left(\cos\left(\tfrac{4\pi}{3}\right) + i\sin\left(\tfrac{4\pi}{3}\right)\right)$$
$$2\left(\cos\left(\tfrac{11\pi}{6}\right) + i\sin\left(\tfrac{11\pi}{6}\right)\right)$$

DeMoivre's Theorem.

$z = 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

principle root's angle is $\frac{\pi}{3} \div 4 = \frac{\pi}{12}$

$\sqrt[4]{z} = 2\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$

Increment is $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} = \frac{6\pi}{12}$

$\left.\begin{array}{l} \frac{\pi}{12} + \frac{6\pi}{12} = \frac{7\pi}{12} \\ \frac{13\pi}{12} \\ \frac{19\pi}{12} \end{array}\right\}$ This fixes it.

The other 4th roots

$2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$2\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right)$

$2\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right)$