Factor Theorem

$f(x)$ a polynomial of degree n & $f(c)=0 \Longrightarrow$

$f(x)=(x-c)P(x)$, where the depressed polynomial has degree $n-1$.

Fundamental Theorem of Algebra

f a polynomial of degree $n \geq 0 \Longrightarrow$

f has at least one zero.

Linear Factorization Theorem

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

$\Longrightarrow f(x) = a_n (x-c_1)(x-c_2)\cdots(x-c_n)$

If there are repeated roots:

c_1, m_1

$c_2, m_2 \quad \Longrightarrow f(x) = a_n (x-c_1)^{m_1}(x-c_2)^{m_2}\cdots(x-c_k)^{m_k}$,

$\vdots$

$c_k, m_k \quad$ where $m_1+m_2+\dots+m_k = n$.

$f(x) = 3(x-2)^3(x+5)^2(x-2+i)^3(x-2-i)^3$

is degree $n=11$

$c_1=2, \ m_1=3$

$c_2=-5, \ m_2=2$

$c_3=2-i, \ m_3=3$

$c_4=2+i, \ m_4=3$

Doing Quadratic Formula the Right Way.

Solve $4x^2-8x+7$

$a=4, b=-8, c=7$

$b^2-4ac = 8^2-4(4)(7) = 64-112 = -48$

$4 \cdot 16$
7
112
-64
48

$\sqrt{-48}$

$2 | 48$
$2 | 24$
$2 | 12$
$2 | 6$
3

$= 4\sqrt{3}\, i$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{8 \pm 4i\sqrt{3}}{2(4)} = \frac{4(2 \pm i\sqrt{3})}{4(2)} = \boxed{\frac{2 \pm i\sqrt{3}}{2} = x}$$

$$x = \frac{8 \pm \sqrt{8^2-4(4)(7)}}{2(4)}$$

Shows bad training & technique

Equations that are quadratic in form.

Solve $x^4+2x^2=8$

way I see them

$(x^2+4)(x^2-2) = (x^2-(2i)^2)(x^2-(\sqrt{2})^2)$

$= (x-2i)(x+2i)(x-\sqrt{2})(x+\sqrt{2}) \Rightarrow x = \pm 2i, \pm\sqrt{2}$

Another (more formal) way

Let $u = x^2 \Rightarrow$

$u^2+2u-8=0 \Rightarrow$

$(u+4)(u-2)=0$

$u+4=0 \quad u-2=0$

$u=-4=x^2 \quad u=2=x^2$

etc.

$a=1, b=2, c=-8$

$b^2-4ac = 2^2-4(1)(-8)$

$= 4+32 = 36$

$\sqrt{36} = 6$

$x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{-2\pm 6}{2}$ ↗ $\frac{4}{2}=2$ ↘ $\frac{-8}{2}=-4$

$u=-4, 2 = x^2$

$x^2=-4$

$x = \pm\sqrt{-4} = \pm 2i$

$x^2=2 \Rightarrow$

$x = \pm\sqrt{2}$

u^2+2u-8

$= u^2+2u+1^2-1-8$

$= (u+1)^2-9 \overset{\text{SET}}{=} 0$

$(u+1)^2=9$

$u+1 = \pm\sqrt{9} = \pm 3$

$u = -1\pm 3$ ↗ 2 ↘ -4

$-1\pm\sqrt{9} = -1\pm 3$ ↗ 2 ↘ -4

$x^2=2 \qquad x^2=-4$, etc.

I suck at factoring, but am asked to factor,

I use a sledgehammer:

Factor $28x^2 - 37x - 110$

$b^2 - 4ac = 37^2 - 4(28)(-110) = 13689$

$\cancel{= 2489}$

$\sqrt{13689} = 117$

$\cancel{\sqrt{2489}}$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{37 \pm 117}{2(28)} =$

$\dfrac{154}{56} = \dfrac{77}{28} = \dfrac{11}{4}$

$\dfrac{-80}{56} = \dfrac{-40}{28} = \dfrac{-10}{7}$

$x = \dfrac{11}{4}, -\dfrac{10}{7}$

$f(x) = 28\left(x - \dfrac{11}{4}\right)\left(x + \dfrac{10}{7}\right)$

$= 7 \cdot 4 \left(x - \dfrac{11}{4}\right)\left(x + \dfrac{10}{7}\right) = (4x - 11)(7x + 10)$

Factor by Grouping (Recognize when you have one.)

$5:2$

$2x^3 - 5x^2 + 22x - 55$

$= x^2(2x-5) + 11(2x-5)$

$= (2x-5)(x^2+11)$

$= (2x-5)(x - \sqrt{11}\,i)(x + \sqrt{11}\,i)$

$2x-5=0$

$2x=5$

$x=\frac{5}{2}$

$x^2+11=0$

$x^2=-11$

$x=\pm\sqrt{-11}=\pm\sqrt{11}\,i$

$x^2+11=0$

$a=1, b=0, c=11$

$b^2-4ac=0^2-4(1)(11)$

$=-44$

$\sqrt{-44}=i\cdot 2\sqrt{11}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{\pm\sqrt{-44}}{2}$

$=\pm\frac{2i\sqrt{11}}{2}=\pm\sqrt{11}\,i$

Use the given zero to find all the zeros of the function

$2x^3-11x^2+26x-21$

BOOK WAY: $c=2-\sqrt{3}i$ is a zero.

$(x-2-\sqrt{3}i)(x-2+\sqrt{3}i)$

$-(-3)$

$= x^2-2x+\cancel{\sqrt{3}ix}-2x+4-\cancel{2\sqrt{3}i}-\cancel{\sqrt{3}ix}+\cancel{2\sqrt{3}i}-(\sqrt{3}i)^2$

$= x^2-4x+7$

$$
\begin{array}{r|l}
 & 2x \quad -3 \\
x^2-4x+7 & 2x^3-11x^2+26x-21 \\
 & -(2x^3-8x^2+14x) \\
 & -3x^2+12x-21 \\
 & -(-3x^2+12x-21) \\
 & 0
\end{array}
$$

$\frac{2x^3}{x^2}=2x$

$\frac{-3x^2}{x^2}=-3$

$(2x-3)(x^2-4x+7)$

$\boxed{=(2x-3)(x-2+\sqrt{3}i)(x-2-\sqrt{3}i) \text{ FACTORED}}$

$\boxed{\text{zeros: } x=2\pm\sqrt{3}i, \frac{3}{2}}$

SECOND WAY:

$2x^3 - 11x^2 + 26x - 21$

$$\begin{array}{r|rrrr} 2-\sqrt{3}i & 2 & -11 & 26 & -21 \\ & & 4-2\sqrt{3}i & -20+3\sqrt{3}i & 21 \\ \hline 2+\sqrt{3}i & 2 & -7-2\sqrt{3}i & 6+3\sqrt{3}i & 0 \\ & & 4+2\sqrt{3}i & -6-3\sqrt{3}i & \\ \hline & 2 & -3 & 0 & \end{array}$$

So $\boxed{x = 2 \pm \sqrt{3}\, i,\ \frac{3}{2}}$

$(-7-2\sqrt{3}i)(2-\sqrt{3}i) = -14 + 7\sqrt{3}i - 4\sqrt{3}i + 2\sqrt{3}\sqrt{3}i^2$

$= -14 + 3\sqrt{3}i - 6$

Screwed up, somewhere!

$= -20 + 3\sqrt{3}i$

$(34+3\sqrt{3}i)(2-\sqrt{3}i) = 68 - 34\sqrt{3}i + 6\sqrt{3}i + 9$

$4+3) = 21$

$3(2+\sqrt{3}i)(2-\sqrt{3}i) = 3(2^2 + \sqrt{3}^2) = 31$

THIRD WAY: RATIONAL ZEROS THEOREM TO FIND THE REAL ROOT.

$2x^3 - 11x^2 + 26x - 21$

$\pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}, \pm 7, \pm \frac{7}{2}, \pm 21, \pm \frac{21}{2}$

Guess....

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -11 & 26 & -21 \\ & & 3 & -12 & 21 \\ \hline & 2 & -8 & 14 & 0 \end{array}$$

$\Rightarrow \boxed{x = \frac{3}{2}}$ & depressed polynomial

is $2x^2 - 8x + 14 \overset{\text{SET}}{=} 0$

$\Rightarrow x^2 - 4x + 7 = x^2 - 4x + 2^2 - 4 + 7$

$= (x-2)^2 + 3 = 0$

$\boxed{x = 2 \pm \sqrt{3}\, i}$

$x = \frac{3}{2},\ 2 \pm \sqrt{3}\, i$