Today §4.1–4.3

Complex #s

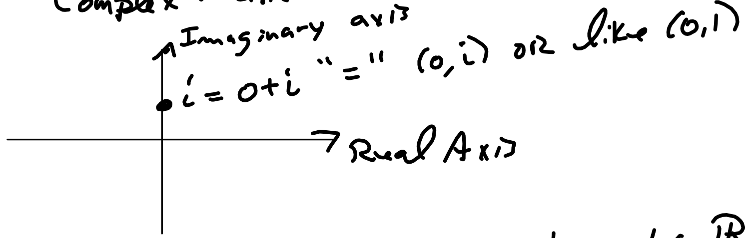
Recall: $x^2+1=0$ has 2 solutions!

$x^2=-1$

$\sqrt{x^2} = |x| = \sqrt{-1} \overset{\text{DEFINE}}{=} i$ = the imaginary unit.

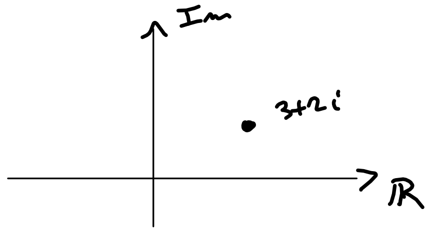
$\boxed{x = \pm i}$

Complex Plane:



$z = a+bi$ is a complex #, with $a, b \in \mathbb{R}$, $b \neq 0$ (to make it not real)

$3+2i$



Sum

$z_1 + z_2 = (a+bi) + (c+di) = (a+c) + (b+d)i$

$(3+2i) + (7-5i) = 10-3i$

Foil

$z_1 z_2 = (a+bi)(c+di) = ac + adi + bci + bdi^2$

$= ac - bd + (ad+bc)i$

$z_1 = (3+2i)$

$z_2 = 7-5i$

$\Rightarrow z_1 z_2 = (3+2i)(7-5i) = 21 + 14i - 15i - 10i^2$

$= 21 + 10 - i = \boxed{31 - i = z_1 z_2}$

Something Special:

$\bar{z}$ = conjugate of z = $a - bi$

Real Part Imaginary Part

$z + \bar{z} = 2a = a+bi + a-bi$

$z\bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 - (-b^2) = a^2+b^2$

Notice it's real.

The complex number field is, in fact a field.

That means

$\exists\ 0 \ni z+0 = 0+z = z$ Additive Identity

$\exists\ 1 \ni z \cdot 1 = 1 \cdot z = z$ Multiplicative Identity

$0 \cdot z = 0 = z \cdot 0$

$z + -z = 0$ Additive Inverse

$z \cdot \frac{1}{z} = 1$ Multiplicative inverse

If z_1, z_2 are in $\mathbb{C} = \{ a+bi \mid a,b \in \mathbb{R} \}$

$\Rightarrow z_1 + z_2 \in \mathbb{C}$ Closure Under Addition

& $z_1 z_2 \in \mathbb{C}$ Closure under Multiplication.

$$\frac{2+5i}{1-3i} = \frac{z_1}{z_2} = (z_1)(z_2^{-1})$$

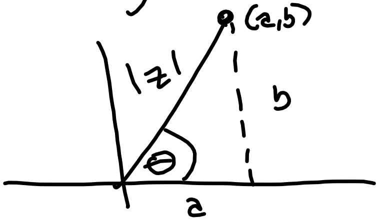
$$\left(\frac{2+5i}{1-3i}\right)\left(\frac{1+3i}{1+3i}\right) = \frac{2+6i+5i-15}{1^2+3^2} = \frac{-13+11i}{10} = -\frac{13}{10} + \frac{11}{10}i$$

"Write $\frac{2+5i}{1-3i}$ in the form $a + b\,i$

Recall $\bar{u} = \langle u_1, u_2 \rangle = \underbrace{\|\bar{u}\|}_{\text{Its Magnitude}} \underbrace{\left\langle \frac{u_1}{\|\bar{u}\|}, \frac{u_2}{\|\bar{u}\|} \right\rangle}_{\text{Unit Vector in the direction of } \bar{u}.}$

$= \|\bar{u}\| \langle \cos\theta, \sin\theta \rangle$, where θ = the (an) argument (Direction Angle) of $\bar{u}$.

Trigonometric Form of $z = a+bi$



$\theta = \arctan\left(\frac{b}{a}\right)$

Modulus of $z = |z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

Similar to vectors $\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}}$

$z = a+bi = |z|(\cos\theta + i\sin\theta)$

$= r(\cos\theta + i\sin\theta)$

Multiplying

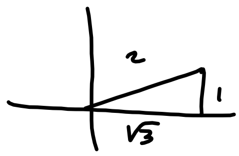
z by a complex # of length (modulus) $r=1$, it amounts to a rotation of z counterclockwise by an amount θ, where $\theta = Arg(w)$

$w = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$ is of length 1

Let $z = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 5i$

What happens when you multiply z by w?

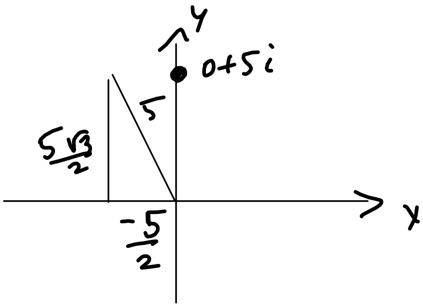
$5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$



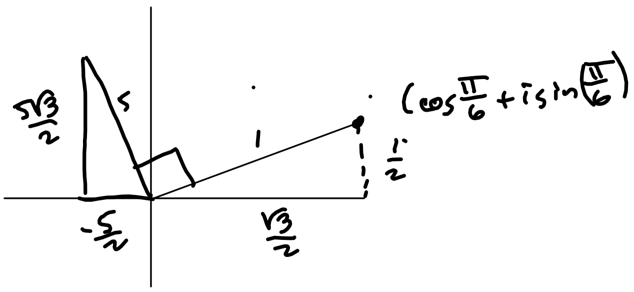
$= 5(0+i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 5\left(i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right) = 5\left(\frac{\sqrt{3}}{2}i + \frac{1}{2}i^2\right) = 5\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$



See how it rotated by $\frac{\pi}{6} = 30^\circ$ when I multiplied it by w?



More formally:

$z = r_1(\cos\theta_1 + i\sin\theta_1)$

$w = r_2(\cos\theta_2 + i\sin\theta_2)$

$zw = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$!

Multiply moduli & add arguments!!

Review of College Algebra's Theorems about (zeros of) Polynomials.

Fundamental Theorem of Algebra - Any polynomial of degree $n \geq 1$ **has at least 1 complex zero.**

Factor Theorem : If c is a zero of a polynomial $P(x)$, then $x-c$ is a factor of $P(x)$

$P(x) = (x-c)(Q(x))$ where degree of $Q(x)$ is one less than the degree of $P(x)$.

$= (x-c)(x-d)(R(x))$

$= a(x-c)(x-d)\cdots(x-k)$, where a = the leading coefficient of $P(x)$

$P(x) = 6x^2 - 11x - 10$ has zeros $x = -\frac{2}{3}, \frac{5}{2} \Longrightarrow$

$P(x) = 6(x+\frac{2}{3})(x-\frac{5}{2})$

$(3x+2)(2x-5)$

$= 6x^2 - 11x - 10$

$x = -\frac{2}{3}, \frac{5}{2}$ are zeros

Conjugate Pairs Theorem:

If the coefficients of $P(x)$ are real & $a+bi$ is a zero, then so is its conjugate $a-bi$.

$P(x) = 30x^4 - 122x^3 + 146x^2 + 38x - 60$

has a complex zero $x = 2+i$. Factor $P(x)$ completely

Fact: $x = 2-i$ is also a zero.

This means $(x-(2+i))(x-(2-i))$ divides into $P(x)$ cleanly.

$= x^2 - (2-i)x - (2+i)x + (2+i)(2-i)$

$= 2x^2 - 2x + ix - 2x - ix + (2^2 + i^2)$

$= 2x^2 - 4x + 5 = d(x) =$ divisor :

$\frac{30x^4}{2x^2} = 15x^2$

$$
\begin{array}{r}
15x^2 - 31x \\
2x^2-4x+5 \,\big)\, 30x^4 - 122x^3 + 146x^2 + 38x - 60 \\
-(30x^4 - 60x^3 + 75x^2) \\
\hline
-62x^3 + 71x^2 + 38x - 60 \\
-(-62x^3 - 124x^2 - 155x) \\
\hline
195x^2 + 493 - 60
\end{array}
$$

$\frac{-62x^3}{2x^2} = -31x$

I messed up the long division, somehow.

$P(x) = \underbrace{(x^2 - 4x + 5)}_{\text{irreducible (over } \mathbb{R}) \text{ quadratic}}(30x^2 - 2x - 12)$

So we find zeros of the depressed polynomial

$30x^2 - 2x - 12 = 0 \Rightarrow$

$15x^2 - x - 6$ $\qquad\qquad 3 \cdot 5 \cdot 3 \cdot 2$

$= 15x^2 - 10x + 9x - 6$

$= 5x(3x-2) + 3(3x-2)$

$= (3x-2)(5x+3) \longrightarrow$

$P(x) = (x-(2-i))(x-(2+i))(3x-2)(5x+3)$

is now split into linear factors

Take your time to be precise in the long division step.