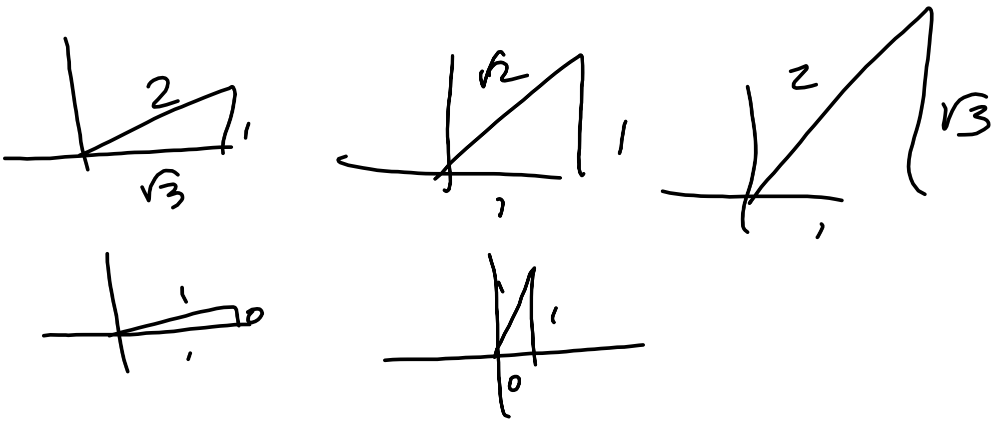
People aren't drawing pictures.

People are still laboring under unit-circle nonsense instead of just learning 3 triangles, 2 degenerate triangles and drawing pictures.

You're making a lot more thought and work for yourself by resisting my teachings.

If you haven't, then take a 1/2-hour and master the 3 triangles in quadrant 1. The rest is pictures and logic. You guys are thinking way too hard, and putting a hard ceiling over your learning.

I teach so that there's *no ceiling*.



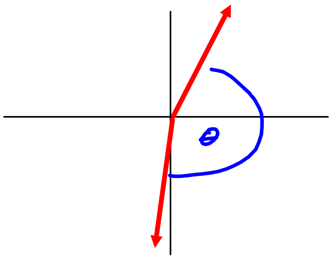
Dot product:

$\bar{u} = \langle u_1, u_2 \rangle$, $\bar{v} = \langle v_1, v_2 \rangle$

$\Rightarrow \bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2$

$\bar{u} = \langle 6, 11 \rangle$, $\bar{w} = \langle -2, -12 \rangle$

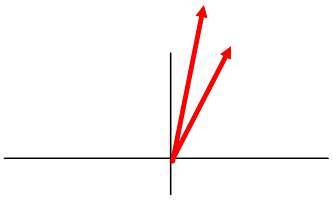
$\Rightarrow \bar{u} \cdot \bar{v} = -12 - 132 = -144$



Almost "parallel" and opposite

$90^\circ < \theta < 180^\circ$

$\bar{u} = \langle 6, 11 \rangle$, $\bar{v} = \langle 2, 12 \rangle$



$\bar{u} \cdot \bar{v} = 12 + 132 = 144$

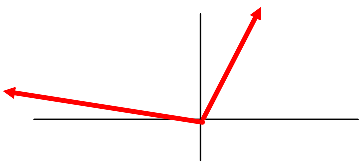
sort of parallel. Sort of in same direction.

$0 < \theta < 90^\circ$

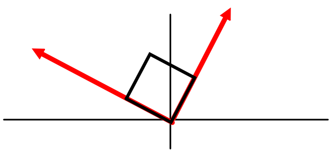
$\bar{u} = \langle 6, 11 \rangle$, $\bar{v} = \langle -12, 2 \rangle$

$\bar{u} \cdot \bar{v} = -72 + 22 = -50$

$90^\circ < \theta < 180^\circ$



$\bar{u} = \langle 6, 11 \rangle$, $\bar{v} = \langle -11, 6 \rangle$



$\bar{u} \cdot \bar{v} = -66 + 66 = 0$

$\theta = 90^\circ$!

Dot product is a lot like cosine.

In fact, dot product is related to cosine in a very specific way!

FACTS about Dot product:

$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$ Commutative property

$\bar{u} \cdot \bar{u} = \langle u_1, u_2 \rangle \cdot \langle u_1, u_2 \rangle = u_1^2 + u_2^2 = \|\bar{u}\|^2$,

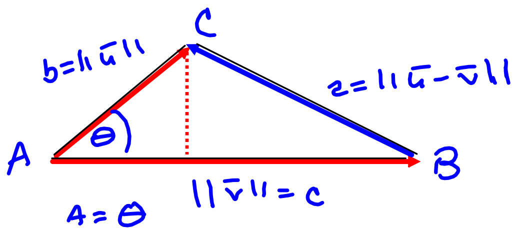
since $\|\bar{u}\| = \sqrt{u_1^2 + u_2^2}$!

$\bar{u} \cdot \bar{u} = \|\bar{u}\|^2$

We derive a formula that gives cosine entirely in terms of dot products.

This is the fundamental connection between cosine and dot products.

$\bar{u}-\bar{v}$



Law of cosines:

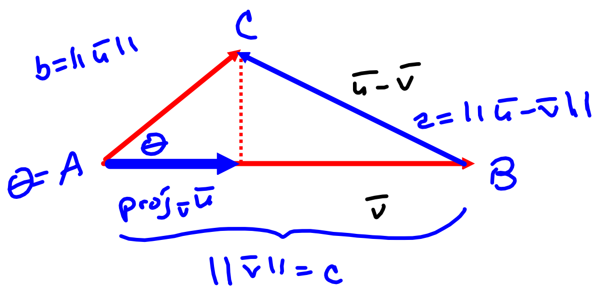
$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{\|\bar{u}\|^2+\|\bar{v}\|^2-\|\bar{u}-\bar{v}\|^2}{2\|\bar{u}\|\|\bar{v}\|}$$

$$= \frac{\|\bar{u}\|^2+\|\bar{v}\|^2-\left((\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v})\right)}{2\|\bar{u}\|\|\bar{v}\|}$$

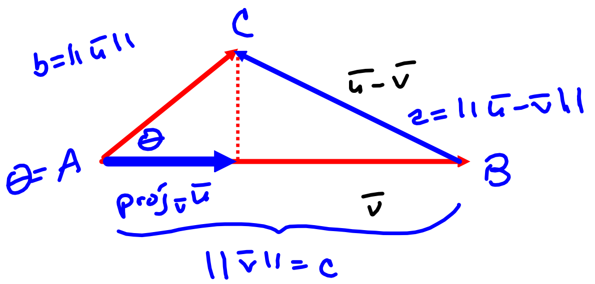
$$= \frac{\bar{u}\cdot\bar{u}+\bar{v}\cdot\bar{v}-(\bar{u}\cdot\bar{u}-2\bar{u}\cdot\bar{v}+\bar{v}\cdot\bar{v})}{2\|\bar{u}\|\|\bar{v}\|}$$

$$= \frac{2\bar{u}\cdot\bar{v}}{2\|\bar{u}\|\|\bar{v}\|} = \boxed{\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \cos\theta}$$

PROJECTION of ubar ONTO vbar



$\text{Proj}_{\bar{v}}\bar{u}$ = Projection of $\bar{u}$ onto $\bar{v}$.



$\|\text{proj}_{\bar{v}} \bar{u}\| =$

$$\frac{\|\text{proj}_{\bar{v}} \bar{u}\|}{\|\bar{v}\|} = |\cos\theta|$$

$$= \begin{cases} -\cos\theta & \text{when } 90^\circ < \theta < 180^\circ \\ \cos\theta & \text{when } 0 < \theta < 90^\circ \end{cases}$$

$$\Longrightarrow \|\text{proj}_{\bar{v}} \bar{u}\| = \|\bar{u}\| \, |\cos\theta|$$

$\text{proj}_{\bar{v}} \bar{u} = \|\text{proj}_{\bar{v}} \bar{u}\| \dfrac{1}{\|\bar{v}\|} \bar{v}$ almost.

Case i: $\theta < 90^\circ$:

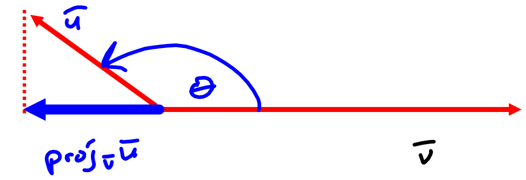
$$\text{proj}_{\bar{v}} \bar{u} = \|\text{proj}_{\bar{v}} \bar{u}\| \frac{1}{\|\bar{v}\|} \bar{v}$$

case ii: $\theta > 90^\circ$

$\text{proj}_{\bar{v}} \bar{u} = -\|\text{proj}_{\bar{v}} \bar{u}\| \dfrac{1}{\|\bar{v}\|} \bar{v}$, b/c $\text{proj}_{\bar{v}} \bar{u}$ will point in the opposite direction from $\bar{v}$.

$b = \|\bar{u}\|$

Case ii: $\theta > 90^\circ$



Now, in case i, $0 < \theta < 90^\circ$, $\cos\theta > 0$ &

$proj_{\bar{v}} \bar{u}$ is pointing in the same direction as $\bar{v}$ and

has magnitude

$$\|proj_{\bar{v}} \bar{u}\| = \big|\,\|\bar{u}\| \cos\theta\,\big|$$

$$= \|\bar{u}\| |\cos\theta| = \|\bar{u}\| \cos\theta$$

In case ii, $90^\circ < \theta < 180^\circ$

$\longrightarrow$ $\|proj_{\bar{v}} \bar{u}\| = \|\bar{u}\|\, |\cos\theta| = -\|\bar{u}\| \cos\theta.$

and $proj_{\bar{v}} \bar{u}$ is in the direction of $-\bar{v}$

Case (i):

$$\|\bar{u}\| \cos\theta \frac{1}{\|\bar{v}\|} \bar{v}$$

case (ii) $-\|\bar{u}\|\, |\cos\theta| \frac{1}{\|\bar{v}\|} \bar{v} = -\|\bar{u}\| (-\cos\theta) \frac{1}{\|\bar{v}\|} \bar{v}$

$$= \|\bar{u}\| \cos\theta \frac{1}{\|\bar{v}\|} \bar{v}$$

$\therefore$ $proj_{\bar{v}} \bar{u} = \|\bar{u}\| \cos\theta \frac{1}{\|\bar{v}\|} \bar{v}$

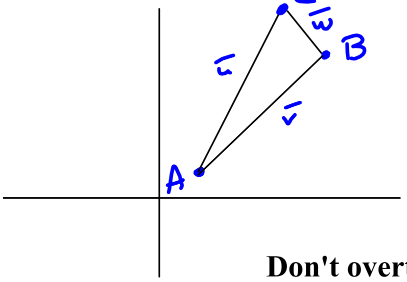
$$= \cancel{\|\bar{u}\|} \frac{\bar{u}\cdot\bar{v}}{\cancel{\|\bar{u}\|}\|\bar{v}\|} \cdot \frac{1}{\|\bar{v}\|} \bar{v}$$

$$= \boxed{\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2} \bar{v} = proj_{\bar{v}} \bar{u}}$$

$$\boxed{\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\| \|\bar{v}\|}}$$

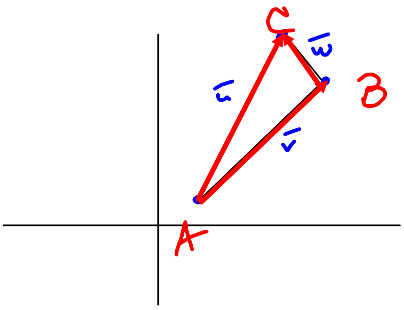
Use vectors to find the interior angles of the triangle with the given vertices.

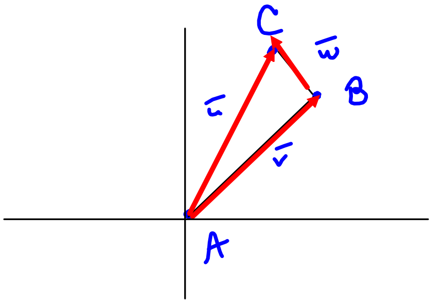
(3,4), (11,12), and (10,13) = A, B, & C, respectively



$$\boxed{\begin{aligned} \overrightarrow{AC} &= \langle 10-3,\ 13-4\rangle = \vec{u} = \langle 7, 9\rangle \\ \overrightarrow{AB} &= \langle 11-3,\ 12-4\rangle = \vec{v} = \langle 8, 8\rangle \\ \overrightarrow{BC} &= \langle 10-11,\ 13-12\rangle = \vec{w} = \langle -1, 1\rangle \end{aligned}}$$

Don't overthink it, Steve. Just put the vectors butt-to-butt to find the angle.





$\|\vec{u}\| = \sqrt{49+81}$

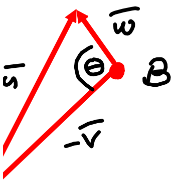
$= \sqrt{130}$

$\|\vec{v}\| = \sqrt{2 \cdot 64} = 8\sqrt{2}$

$\|\vec{w}\| = \sqrt{2}$

$$\cos A = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{56+72}{\sqrt{130}\cdot 8\sqrt{2}} = \frac{128}{8\sqrt{260}} \approx 7.1250163489^\circ$$

Now, to find B, Note B = angle between $\vec{w}$ and $-\vec{v}$

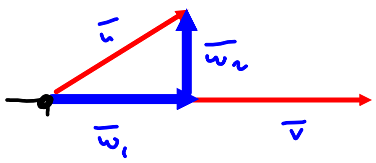


$$\cos\theta = \frac{-\vec{v}\cdot\vec{w}}{\|-\vec{v}\|\|\vec{w}\|} = \frac{-\langle 8,8\rangle\cdot\langle -1,1\rangle}{(8\sqrt{2})(\sqrt{2})} = \frac{-(-8+8)}{\sim} = 0$$

$$\rightarrow \boxed{B = 90^\circ = \frac{\pi}{2}}$$

$$\therefore C = 180^\circ - A - B \approx \boxed{82.8749836511^\circ \approx C}$$

Building orthogonal components.



GRAM-SCHMIDT ORTHOGONALIZATION

we decompose $\bar{u} = \bar{w}_1 + \bar{w}_2$ = The sum of a vector $\bar{w}_1$ in the direction of $\bar{v}$ (i.e. $\text{proj}_{\bar{v}}\bar{u} = w_1$) and $\bar{w}_2 = \bar{u} - \bar{w}_1$, that is.

$$\bar{u} = \text{proj}_{\bar{v}}\bar{u} + (\bar{u} - \text{proj}_{\bar{v}}\bar{u})$$
$$= \bar{w}_1 + \bar{w}_2 .$$

Note $\bar{w}_1 \perp \bar{w}_2$:

$$\bar{w}_1 \cdot \bar{w}_2 = (\text{proj}_{\bar{v}}\bar{u}) \cdot (\bar{u} - \text{proj}_{\bar{v}}\bar{u})$$

$$= \left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}\right)\cdot\bar{u} - \left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}\right)\cdot\left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}\right)$$

$$= \frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}\cdot\bar{u} - \frac{(\bar{u}\cdot\bar{v})^2}{\|\bar{v}\|^4}\bar{v}\cdot\bar{v}$$

$$= \frac{(\bar{u}\cdot\bar{v})^2}{\|\bar{v}\|^2} - \frac{(\bar{u}\cdot\bar{v})^2}{\|\bar{v}\|^4}\|\bar{v}\|^2$$

$$= \frac{(\bar{u}\cdot\bar{v})^2 - (\bar{u}\cdot\bar{v})^2}{\|\bar{v}\|^2} = 0 \checkmark$$

Two mutually orthogonal vectors of length 1:

The canonical basis for $\mathbb{R}^2$

$\bar{e}_1 = \bar{i} = \langle 1, 0 \rangle, \ \bar{j} = \langle 0, 1 \rangle$

You can write any vector $\bar{u}$ as a linear combination of $\bar{i}$ & $\bar{j}$:

$\bar{u} = \langle 3, -7 \rangle \longrightarrow$

$\bar{u} = 3\bar{i} - 7\bar{j}$

$= 3\langle 1, 0 \rangle - 7\langle 0, 1 \rangle$

Recall $\bar{w}_1$ & $\bar{w}_2$ from before

$\vec{u} = \langle 7,5 \rangle$

$\bar{v} = \langle 10,1 \rangle$

write $\bar{u}$ as the sum of 2 orthogonal vectors $\bar{w}_1$ & $\bar{w}_2$ such that $\bar{w}_1 \parallel \bar{v}$ & $\bar{w}_2 \perp \bar{v}$.

$\bar{w}_1 = \text{proj}_{\bar{v}}\bar{u} = \dfrac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}$

$= \dfrac{\langle 7,5\rangle\cdot\langle 10,1\rangle}{10^2+1^2}\langle 10,1\rangle$

$= \dfrac{70+5}{101}\langle 10,1\rangle = \boxed{\dfrac{75}{101}\langle 10,1\rangle = \bar{w}_1}$

$\bar{w}_2 = \bar{u} - \bar{w}_1 = \langle 7,5\rangle - \dfrac{75}{101}\langle 10,1\rangle$

$= \left\langle \dfrac{707-750}{101}, \dfrac{505-75}{101} \right\rangle$

$= \boxed{\left\langle \dfrac{-43}{101}, \dfrac{430}{101} \right\rangle = \bar{w}_2}$

Check:

$\bar{w}_1 + \bar{w}_2 = \dfrac{75}{101}\langle 10,1\rangle + \left\langle \dfrac{-43}{101}, \dfrac{430}{101}\right\rangle$

$= \left\langle \dfrac{750-43}{101}, \dfrac{75+430}{101}\right\rangle$

$= \left\langle \dfrac{707}{101}, \dfrac{505}{101}\right\rangle = \langle 7,5\rangle = \bar{u}$ ✓

$\bar{w}_1\cdot\bar{w}_2 = \dfrac{75}{101}\langle 10,1\rangle\cdot\left(\dfrac{1}{101}\langle -43,430\rangle\right)$

$= \dfrac{75}{101^2}\left(\langle 10,1\rangle\cdot\langle -43,430\rangle\right)$

$= \dfrac{75}{101^2}\left(-430+430\right) = 0$ ✓

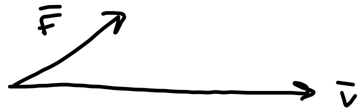
FACT:

$\cos 90^0 = 0$

$\bar{u}\cdot\bar{v} = 0 \iff \bar{u}$ is orthogonal to $\bar{v}$

$(\bar{u} \perp \bar{v})$

Work = Force Times Distance



= (Component of Force in direction of Travel)
Times (Distance Traveled in direction of travel)

$$= \|\mathrm{proj}_{\bar v}\bar F\|\,\|\bar v\|$$

$$= \boxed{\left\| \frac{\bar F\cdot\bar v}{\|\bar v\|^2}\bar v \right\|}\,\|\bar v\|$$

$$= \frac{|\bar F\cdot\bar v|\,\|\bar v\|\,\|\bar v\|}{\|\bar v\|^2} = |\bar F\cdot\bar v|$$

$$\|c\bar v\| = |c|\,\|\bar v\|$$

$$\left\|\frac{\bar F\cdot\bar v}{\|\bar v\|^2}\bar v\right\|$$

$$= \left|\frac{\bar F\cdot\bar v}{\|\bar v\|^2}\right|\|\bar v\|$$

$$= |\bar F\cdot\bar v|\left(\frac{1}{\|\bar v\|^2}\|\bar v\|\right)$$

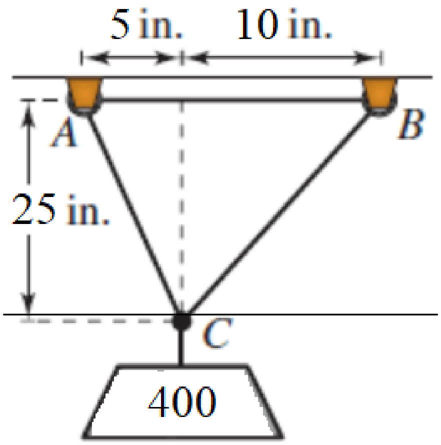
$$=$$

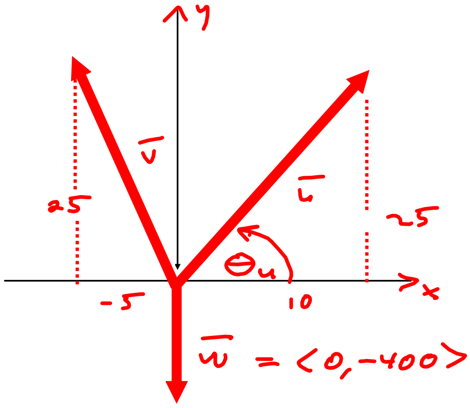
$= \bar F\cdot\bar v$ when $0 < \theta < 90^\circ$

$= -\bar F\cdot\bar v$ " $90^\circ < \theta < 180^\circ$

In the real world, we have $0 < \theta < 90^\circ$ because motion along $\bar v$ is always in the direction of the force.

Hence work done by $\bar F$ in direction of $\bar v$ is simply $\bar F\cdot\bar v$!

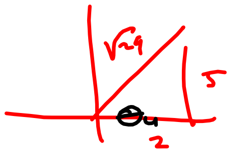




$\bar u$ is no problem. $\bar u = \|\bar u\| \langle \cos\theta_u, \sin\theta_u \rangle$,

where $\theta_u = \arctan\left(\frac{25}{10}\right) \longrightarrow \cos\theta_u = \cos\left(\arctan\left(\frac{5}{2}\right)\right)$

Picture:



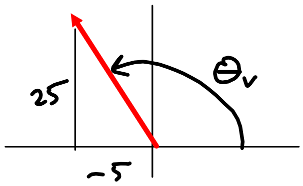
$5^2 + 2^2 = 25 + 4 = 29 \longrightarrow$

$c = \sqrt{29}$

$\therefore \boxed{\cos\theta_u = \frac{2}{\sqrt{29}}}$ & $\boxed{\sin\theta_u = \frac{5}{\sqrt{29}}}$

$\therefore \boxed{\bar u = \|\bar u\| \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle}$

We have to be careful with θ_v:



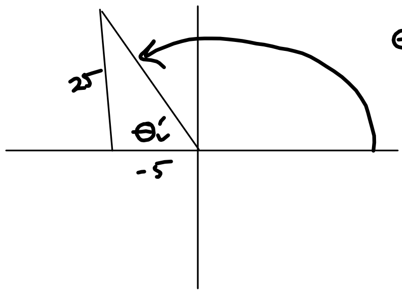
We have to make sure that θ_v ends up in Q II, $\frac{\pi}{2} < \theta_v < \pi$

$(90^\circ < \theta_v < 180^\circ)$

$\arctan\left(\frac{25}{-5}\right) = \arctan(-5) = -\arctan(5)$

has the same reference angle as $\arctan(5)$

That is to say, $\theta_v' = \arctan(5)$

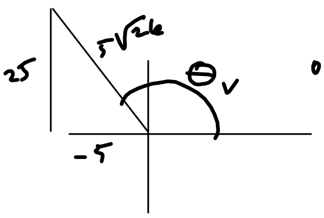


$\theta_v = 180^\circ - \theta_v' = 180^\circ - \arctan\left(\frac{25}{5}\right)$

$= 180^\circ - \arctan(5)$

Now, $\sqrt{5^2 + 25^2} = \sqrt{5^2(1 + 5^2)}$

$= 5\sqrt{26}$



$\therefore \cos(\theta_v) = \frac{-5}{5\sqrt{26}} = -\frac{1}{\sqrt{26}}$

and $\sin(\theta_v) = \frac{25}{5\sqrt{26}} = \frac{5}{\sqrt{26}}$

$\therefore \bar v = \|\bar v\| \langle \cos\theta_v, \sin\theta_v \rangle$

$$\boxed{\bar{u} = \|\bar{u}\| \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle}$$

$$\boxed{\bar{v} = \|\bar{v}\| \left\langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right\rangle}$$

Let $x = \|\bar{u}\|$, $y = \|\bar{v}\|$

We want $\bar{u} + \bar{v} = \langle 0, 400 \rangle$.

This means we want

$$\left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle x + \left\langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right\rangle y = \langle 0, 400 \rangle$$

E1 $\frac{2}{\sqrt{29}} x - \frac{1}{\sqrt{26}} y = 0$ and

E2 $\frac{5}{\sqrt{29}} x + \frac{5}{\sqrt{26}} y = 400$, so

$$\frac{2}{\sqrt{29}} x = \frac{1}{\sqrt{26}} y \longrightarrow$$

$$x = \frac{\sqrt{29}}{2} \cdot \frac{1}{\sqrt{26}} y = \frac{\sqrt{29}}{2\sqrt{26}} y$$

$$\begin{array}{r} 29 \\ \times 26 \\ \hline 174 \\ 58 \\ \hline 754 \end{array}$$

Now bring out E2:

$$\frac{5}{\sqrt{29}} \left(\frac{\sqrt{29}}{2\sqrt{26}} y \right) + \frac{5}{\sqrt{26}} y = 400 \longrightarrow$$

$$\longrightarrow \frac{5}{2\sqrt{26}} y + \frac{5}{\sqrt{26}} y = \frac{5+10}{2\sqrt{26}} y = 400 \longrightarrow$$

$$y = \frac{400(2\sqrt{26})}{15} = \frac{800\sqrt{26}}{15} = \boxed{\frac{160\sqrt{26}}{3} = y = \|\bar{v}\|}$$

& so,

$$x = \frac{\sqrt{29}}{2\sqrt{26}} y = \frac{\sqrt{29}}{2\sqrt{26}} \left(\frac{160\sqrt{26}}{3} \right) = \boxed{\frac{80\sqrt{29}}{3} = x = \|\bar{u}\|}$$