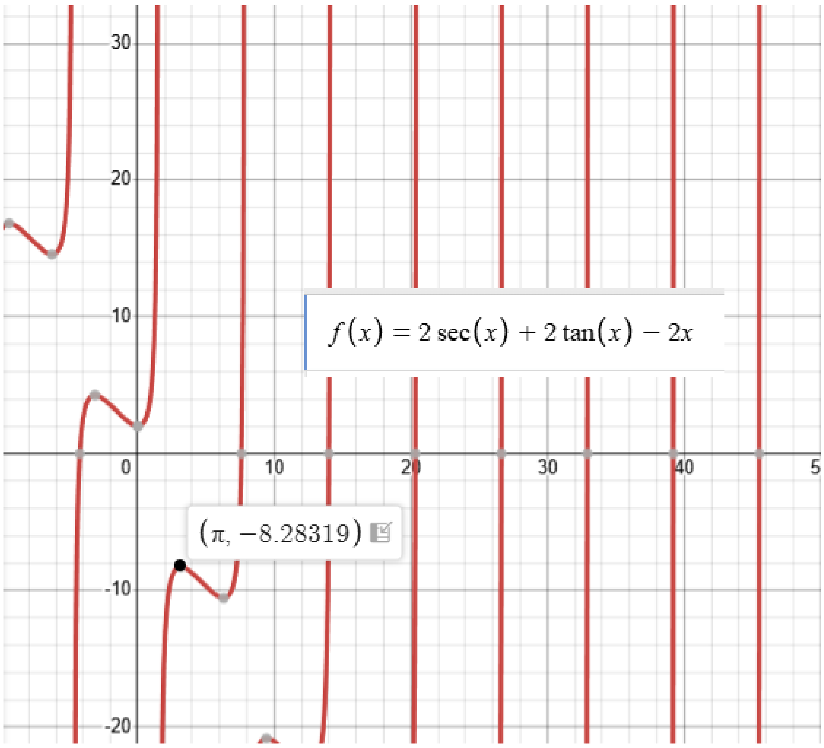
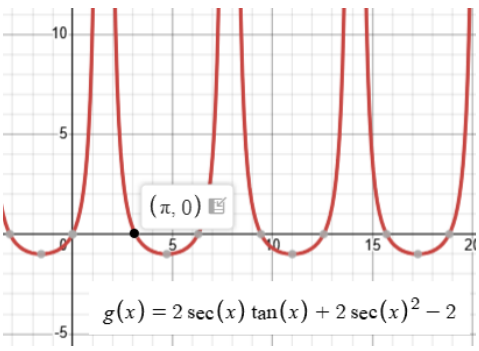
2.3 #44





Section 2.5 Multiple-Angle and Product-to-Sum Formulas

Double-Angle

We proved portions of the following last week on 2/26. **Click Here for 2/26 notees.**

$\sin 2u = 2\sin u \cos u$	$\cos 2u = \cos^2 u - \sin^2 u$
$\tan 2u = \dfrac{2\tan u}{1-\tan^2 u}$	$= 2\cos^2 u - 1$
	$= 1 - 2\sin^2 u$

Solve $\cos 2x + \cos x = 0$.

Cheat Sheet for Trig

$\cos(2x) = 2\cos^2(x) - 1$

$2\cos^2(x) - 1 + \cos(x) = 0$

$2\cos^2(x) + \cos(x) - 1 = 0$

"Quadratic in cosine."

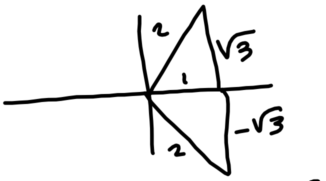
$2v^2 + v - 1 = 0$

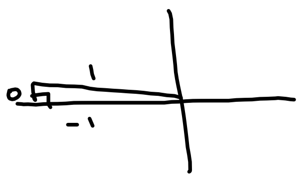
$(2v-1)(v+1) = 2v^2 + 2v - v - 1$ ✓

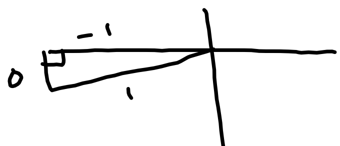
$\longrightarrow 2\cos(x) - 1 = 0$ OR $\cos(x) + 1 = 0$

$2\cos(x) = 1$ $\qquad \cos(x) = -1$

$\cos(x) = \frac{1}{2}$







$x = 60°, 300°$

$= \frac{\pi}{3}, \frac{5\pi}{3}$

$x = 180° = \pi$

$x = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n, \pi + 2\pi n, n \in \mathbb{Z}$

$x \in \{ \omega + 2\pi n \mid \omega = \frac{\pi}{3}, \frac{5\pi}{3}, \pi, n \in \mathbb{Z} \}$

$= \{ \omega + 2\pi n \mid \omega \in \{\frac{\pi}{3}, \frac{5\pi}{3}, \pi\}, n \in \mathbb{Z} \}$

Set-builder answers.

Midterm about a week after we get back from Spring Break.

Old Tests buried in the Homework Videos and Notes for each chapter.

Last semester's Midterm.

Midterm and Final are in-person. Bring a scientific calculator, student ID, pencil/pen

HOR 107 8 am - 6 pm Show Up. Place shuts down at 8 pm. Don't show up any later than 6 pm if you want 2 hours to take it.

Use the conditions below to find $\sin\, 2\theta$, $\cos\, 2\theta$, and $\tan\, 2\theta$

$$\sin\, \theta = \frac{3}{5}, \quad 0 < \theta < \frac{\pi}{2}$$

Power-Reducing Formulas

These are obtained by manipulating the double-angle formulas. Take the principle square root of both sides, relabel things, and you obtain the 1/2-angle formulas, below:

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Half-Angle Formulas

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta = \cos(2\theta)$$

$$-2\sin^2\theta = \cos(2\theta) - 1$$

$$\sin^2\theta = \frac{\cos(2\theta)-1}{-2} = \boxed{\frac{1-\cos(2\theta)}{2} = \sin^2\theta}$$

$$\Rightarrow \sqrt{\sin^2\theta} = \sqrt{\frac{1-\cos(2\theta)}{2}}$$

$$|\sin\theta| = \sqrt{\frac{1-\cos(2\theta)}{2}}$$

$$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$

Let $u = 2\theta$:

$$\boxed{\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos(u)}{2}}}$$

$\frac{1}{2}$- angle.

I don't think these help us a ton in Trig, but they will be very useful in Calculus II.

Product-to-Sum Formulas

$$\sin u \sin v = \tfrac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u-v) + \cos(u+v)]$$

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u+v) - \sin(u-v)]$$

Redundant. Just re-label.

$$\cos(u)\sin(v) = \sin(v)\cos(u) = \sin(w)\cos(z)$$

$$= \tfrac{1}{2}[\sin(w+z) + \sin(w-z)]$$

$$= \tfrac{1}{2}[\sin(v+u) + \sin(v-u)]$$

$$= \tfrac{1}{2}[\sin(v+u) - \sin(u-v)]$$

The following are obtained by re-labeling and manipulating Product-to-Sum Formulas

I don't think I see these coming up more than once every 20 years or so, in my case, and pretty much only in teaching trig. I

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

here's the example for how these help. It doesn't look like anything you couldn't handle with other tools that you should know a lot better than this one.

Example 8 Using a Sum-to-Product Formula

Find the exact value of $\cos 195^\circ + \cos 105^\circ$.

Solution

Use the appropriate sum-to-product formula.

$$\begin{aligned}\cos 195^\circ + \cos 105^\circ &= 2\cos\left(\frac{195^\circ + 105^\circ}{2}\right)\cos\left(\frac{195^\circ - 105^\circ}{2}\right)\\ &= 2\cos 150^\circ \cos 45^\circ\\ &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\end{aligned}$$

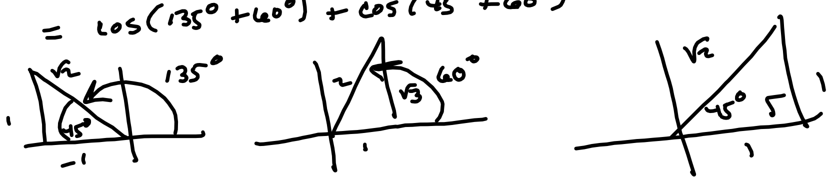
My / Our way:

$105^\circ = 60^\circ + 45^\circ$

$195^\circ = 135^\circ + 60^\circ$

$\therefore \cos(195^\circ) + \cos(105^\circ)$

$= \cos(135^\circ + 60^\circ) + \cos(45^\circ + 60^\circ)$



$= \cos(135^\circ)\cos(60^\circ) - \sin(135^\circ)\sin(60^\circ)$

$+ \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ)$

$= -\frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}$

$= -\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = -\frac{\sqrt{3}}{\sqrt{2}}$ OR $-\frac{\sqrt{6}}{2}$

I get the right answer, here, but I can't say it was as efficient as sum-to-product.

Sum-to-Product can make this one go quickly. But again, it's a lot of baggage for the one task, especially when you have other tools that can do this for you. Writing non-stop without thinking a whole lot beats stopping, thinking, looking through a list of 500 formulas for The One.

Example 9 Solving a Trigonometric Equation

Solve $\sin 5x + \sin 3x = 0$.

Sounds good, Mills. But it really does shorten the work, here.

I'm just not huge on asking these questions at this level. If I'm going to hit you with sum-to-product on a test, I'll TELL you to use it. Otherwise, this is just piling on too many formulas for this level... In my opinion.

Book Solution using sum-to-product nonsense:

Solution

$$\sin 5x + \sin 3x = 0 \quad \text{Write original equation.}$$

$$2\sin\left(\tfrac{5x+3x}{2}\right)\cos\left(\tfrac{5x-3x}{2}\right) = 0 \quad \text{Sum-to-product formula}$$

$$2\sin 4x \cos x = 0 \quad \text{Simplify.}$$

Set the factor $2\sin 4x$ equal to zero. The solutions in the interval $[0, 2\pi)$ are

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

The equation $\cos x = 0$ yields no additional solutions, so the solutions are of the form $x = n\pi/4$, where n is an integer. To confirm this graphically, sketch the graph of $y = \sin 5x + \sin 3x$ as shown below.

I have to admit, this is pretty slick.

Example 9 Solving a Trigonometric Equation

Solve $\sin 5x + \sin 3x = 0$.

$\Rightarrow \sin(2x+3x) + \sin(3x) = 0$

$\rightarrow \sin(2x)\cos(3x) + \sin(3x)\cos(2x) + \sin(3x) = 0$

$\Rightarrow \sin(2x)\cos(2x+x) + \sin(2x+x)\cos(2x) + \sin(2x+x) = 0$

$\Rightarrow \sin(2x)\big(\cos(2x)\sin(x) - \cos(x)\sin(2x)\big) + \big(\sin(2x)\cos(x) + \sin(x)\cos(2x)\big)\cos(2x)$

$+ \sin(2x)\cos(x) + \sin(x)\cos(2x) =$

$= \sin(2x)\cos(2x)\sin(x) - \sin^2(2x)\cos(x) + \sin(2x)\cos(x)\cos(2x) + \sin(x)\cos^2(2x)$

$+ \sin(2x)\cos(x) + \sin(x)\cos(2x)$

$= \sin(2x)\cos(2x)\sin(x) - \sin^2(x)\cos(x)$

Bleah. So much for THAT.!

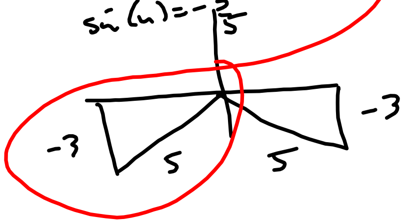
A few minutes later....

Holy crap this got ugly! I bet I could come up with something with this method, but not very elegantly.

FWIW, I don't ask these kinds of questions. My focus is the double-angle, power-reducing, half-angle and angle-sum formulas, and everything I ask can be solved with them.

Find $\sin\left(\frac{u}{2}\right)$, $\cos\left(\frac{u}{2}\right)$, $\tan\left(\frac{u}{2}\right)$, given

$\sin(u) = -\frac{3}{5}$, $\cos(u) < 0$



This tells me we're in quadrant III

$\therefore\ \pi < u < \frac{3\pi}{2}$

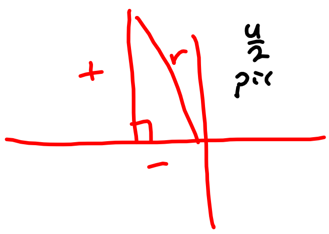
$180° < u < 270°$

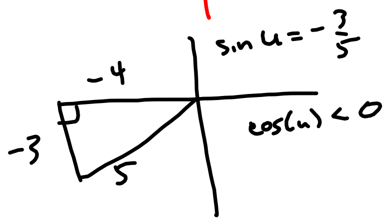
$\longrightarrow\ \frac{\pi}{2} < \frac{u}{2} < \frac{3\pi}{4}$ Q II

$\therefore\ \sin\left(\frac{u}{2}\right) > 0$

$\cos\left(\frac{u}{2}\right) < 0$

$\tan\left(\frac{u}{2}\right) < 0$





$5^2 - 3^2 = 25 - 9 = 16 \rightarrow \sqrt{16} = 4$

Q II $\frac{u}{2}$

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos(u)}{2}} = +\sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}}$$

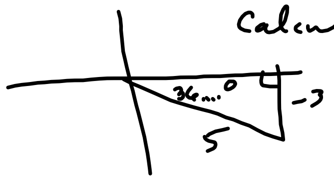
$$= \sqrt{\frac{1+\frac{4}{5}}{2}\cdot\frac{5}{5}} = \sqrt{\frac{5+4}{10}} = \sqrt{\frac{9}{10}} = \boxed{\frac{3}{\sqrt{10}}} \text{ or } \frac{3\sqrt{10}}{10}$$

$$\cos\left(\frac{u}{2}\right) = -\sqrt{\frac{1+\cos(u)}{2}} = \sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}} = \sqrt{\frac{5-4}{10}} = \sqrt{\frac{1}{10}} = \boxed{\frac{1}{\sqrt{10}}} = \frac{\sqrt{10}}{10}$$

$$\tan\left(\frac{u}{2}\right) = \frac{1-\cos(u)}{\sin(u)} = \frac{1-\left(-\frac{4}{5}\right)}{-\frac{3}{5}} = \frac{5+4}{-3} = -\frac{9}{3} = \boxed{-3 = \tan\left(\frac{u}{2}\right)}$$

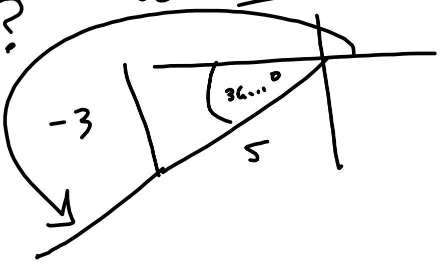
$\arcsin\left(-\frac{3}{5}\right) \approx -36.86989765^\circ$

Calculator sees THIS!



we know it's THIS:

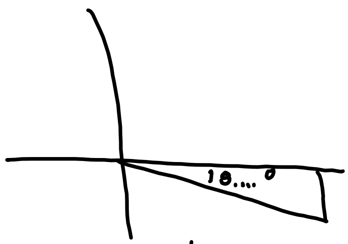
?



? $180^\circ - \arcsin\left(-\frac{3}{5}\right)$

$= 180^\circ + \arcsin\left(\frac{3}{5}\right)$

If all you do is blindly push buttons on your calculator, you're going to think that u/2 is in quadrant IV:



where sine < 0
cosine > 0
tangent < 0

$$\sin(2x)\sin(x) = \cos(x) \Rightarrow$$

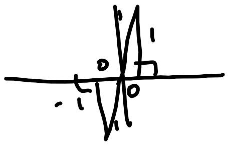
$$2\sin(x)\cos(x)\sin(x) = \cos(x) \Rightarrow$$

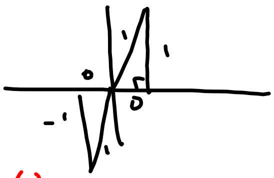
$$\sin^2(x)\cos(x) - \cos(x) = 0 \Rightarrow$$

$$\cos(x)(\sin^2(x) - 1) = 0 \Rightarrow$$

$\cos(x) = 0$ or $\sin^2(x) = 1$

$\sin(x) = \pm 1$





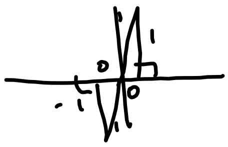
$\emptyset$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

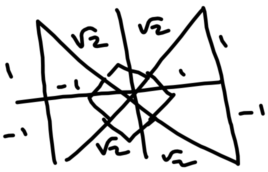
$$2\sin(x)\cos(x)\sin(x) - \cos(x) = 0$$

$$\cos(x)\left(2\sin^2(x) - 1\right) = 0$$

$\cos(x) = 0$ or $\sin^2(x) = \frac{1}{2}$

$\sin(x) = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$





$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$x = \frac{\pi}{2} + n\pi$ is more elegant or $\frac{\pi}{4} + \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

$\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi$ is ok

$x \in \left\{ \frac{\pi}{2} + n\pi, \frac{\pi}{4} + \frac{n\pi}{2} \,\middle|\, n \in \mathbb{Z} \right\}$

$\Rightarrow x = \frac{\pi}{2} + n\pi, \frac{\pi}{4} + \frac{n\pi}{2}, n \in \mathbb{Z}.$