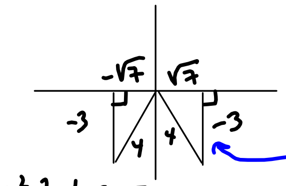
MAT 1420-R11 Written Final, Spring, 2024 Name________________

200 Points Emphasizes Chapters 3, 4, and 6.

Show All Work. Circle Final Answers.

1. (10 pts) Find the value of the 6 trigonometric functions, given that $\sin(u) = -\frac{3}{4}$ and $\frac{3\pi}{2} \le u < 2\pi$.



$4^2 - 3^2 = 16 - 9 = 7$

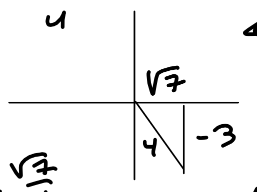
This is the triangle we want

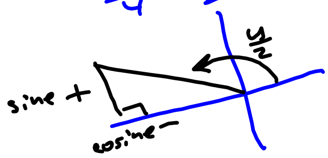
$$\boxed{\begin{aligned} \sin(u) &= -\frac{3}{4} \\ \cos(u) &= \frac{\sqrt{7}}{4} \\ \tan(u) &= -\frac{3}{\sqrt{7}} \text{ OR } -\frac{3\sqrt{7}}{7} \end{aligned}}$$

$$\boxed{\begin{aligned} \csc(u) &= -\frac{4}{3} \\ \sec(u) &= \frac{4}{\sqrt{7}} \text{ OR } \frac{4\sqrt{7}}{7} \\ \cot(u) &= -\frac{\sqrt{7}}{3} \end{aligned}}$$

2. (10 pts) Find $\sin\left(\frac{u}{2}\right)$, $\cos\left(\frac{u}{2}\right)$, and $\tan\left(\frac{u}{2}\right)$, given that $\sin(u) = -\frac{3}{4}$ and $\frac{3\pi}{2} \le u < 2\pi$

$\frac{3\pi}{4} < \frac{u}{2} < \pi$ so QII





$\longrightarrow \cos(u) = \frac{\sqrt{7}}{4}$

Answers may vary.

$$\sin\left(\frac{u}{2}\right) = +\sqrt{\frac{1-\cos(u)}{2}} = \sqrt{\frac{1-\frac{\sqrt{7}}{4}}{2}\cdot\frac{4}{4}} = \sqrt{\frac{4-\sqrt{7}}{8}} = \boxed{\frac{1}{2}\sqrt{\frac{4-\sqrt{7}}{2}}} = \frac{\sqrt{7}-1}{4} \approx .411$$

others may vary

$$\cos\left(\frac{u}{2}\right) = -\sqrt{\frac{1+\cos(u)}{2}} = -\sqrt{\frac{1+\frac{\sqrt{7}}{4}}{2}} = -\sqrt{\frac{4+\sqrt{7}}{8}} = \boxed{-\frac{1}{2}\sqrt{\frac{4+\sqrt{7}}{2}}} = \frac{-\sqrt{7}-1}{4} \approx -.911$$

$$\tan\left(\frac{u}{2}\right) = \frac{\sin\left(\frac{u}{2}\right)}{\cos\left(\frac{u}{2}\right)} = -\sqrt{\frac{4-\sqrt{7}}{4+\sqrt{7}}} = -\sqrt{\frac{16-7}{(4+\sqrt{7})^2}} = -\frac{\sqrt{9}}{4+\sqrt{7}} = -\left(\frac{3}{16-7}\right)(4-\sqrt{7})$$

$$= -\frac{3}{9}(4-\sqrt{7}) = \boxed{\frac{\sqrt{7}-4}{3}} \approx 2.646$$

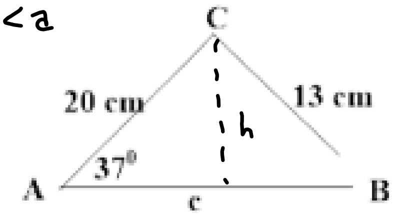
$$\tan\left(\frac{u}{2}\right) = \frac{1-\cos(u)}{\sin(u)} = \frac{1-\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = \frac{4-\sqrt{7}}{-3} = \boxed{\frac{\sqrt{7}-4}{3}} \approx 2.646$$

3. Suppose $A = 37^0$, $a = 13$ cm, and b = 20 cm. (See figure.)

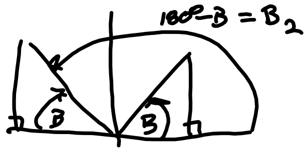
a. (5 pts) Show that there are two solutions to this triangle, before solving the triangle.

$\frac{h}{b} = \sin A \Rightarrow h = 20\sin(37^0) \approx 12.03630047 < a$

$h < a = 13 < 20 = b \Rightarrow \exists$ 2 sol'ns.



b. (5 pts) Find both solutions. For one, B will be acute. For the other, B will be obtuse. Round final answers to 3 decimal places.



B is acute:

$\frac{\sin B}{b} = \frac{\sin A}{a}$

$\sin B = \frac{b \sin A}{a} = \frac{20 \sin 37^0}{13} \approx 0.9258692662$

$B = \arcsin(\sin(B)) \approx 67.79980010^0 \approx B$

$\Rightarrow C = 180^0 - B - A \approx 75.20019993^0$

& finally,

$c = \frac{a \sin C}{\sin A} \approx 20.88468242 \approx c$

$A = 37^0$
$B \approx 67.800^0$
$C \approx 75.200^0$
$a = 13$ cm
$b = 20$ cm
$c \approx 20.885$ cm

The 2nd Solution,

$B_2 = \pi - B$ or BETTER,

$B_2 = 180^0 - B \approx 112.2001999^0$

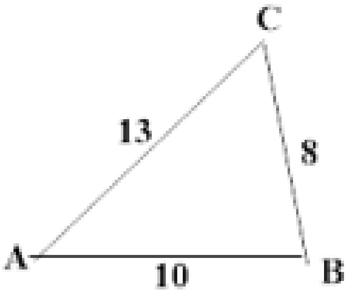
$\Rightarrow$

$C_2 = 180^0 - B_2 - A \approx 30.7998001^0$

$c_2 = \frac{a\sin(C_2)}{\sin(A)} = \frac{13\sin(C_2)}{\sin(37^0)} \approx 11.06073799$ cm $\approx c_2$

$a = 13$ cm
$b = 20$ cm
$c_2 \approx 11.061$ cm
$A_2 = 37^0$
$B_2 \approx 112.200^0$
$C_2 \approx 30.800^0$

4. (10 pts) Suppose $a = 8$, $b = 13$, $c = 10$. Solve the triangle. Round final answers decimal places. See figure.



$a^2 = b^2 + c^2 - 2bc\cos A$

$2bc\cos A = b^2 + c^2 - a^2$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{13^2+10^2-8^2}{2(10)(13)} = \dfrac{269-64}{260} = \dfrac{205}{260}$

$\Rightarrow A = \arccos\left(\frac{205}{260}\right) \approx \boxed{37.958^\circ \approx A} \approx .662$

$\cos C = \dfrac{a^2+b^2-c^2}{2ba} = \dfrac{8^2+13^2-10^2}{2(8)(13)} = \dfrac{64+169-100}{208} = \dfrac{133}{208}$

$\Rightarrow C = \arccos\left(\frac{133}{208}\right) \approx \boxed{50.251^\circ \approx C} \approx .877$

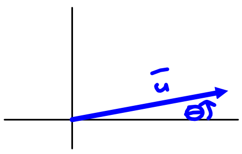
$\Rightarrow B = 180^\circ - A - C \approx \boxed{91.791^\circ \approx B \approx 1.602}$

5. Let $\overline{u} = \langle 7,1 \rangle$ and $\overline{v} = \langle 2,6 \rangle$. Compute or sketch, as directed:

a. (10 pts) $\|\overline{u}\|$ (Use dot product).

$$\|\bar u\| = \sqrt{\bar u \cdot \bar u} = \sqrt{\langle 7,1\rangle \cdot \langle 7,1\rangle} = \sqrt{7^2+1^2} = \sqrt{50} = \boxed{5\sqrt{2} = \|\bar u\|}$$

b. (5 pts) The direction angle for $\overline{u}$. Use degrees. Round final answer to 3 decimal places.



$\bar u = \langle 7,1\rangle$

$\Rightarrow \theta = \arctan\left(\frac{1}{7}\right) \approx 8.130102352$

$\approx \boxed{8.130^\circ \approx \theta}$ $\approx .142$

c. (5 pts) The angle between $\overline{u}$ and $\overline{v}$. Use degrees. Round final answer to 3 decimal places.

$$\cos\theta = \frac{\bar u \cdot \bar v}{\|\bar u\|\|\bar v\|}$$

$\|\bar v\| = \sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$

$\|\bar u\| = 5\sqrt{2}$ $\quad \bar u \cdot \bar v = \langle 7,1\rangle \cdot \langle 2,6\rangle = 14+6=20$

$$\Rightarrow \cos\theta = \frac{20}{(5\sqrt{2})(2\sqrt{10})} \Rightarrow \theta = \arccos\left(\frac{2}{\sqrt{20}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right)$$

$\approx 63.43494883^\circ \approx \boxed{63.435^\circ \approx \theta}$

≈ 1.107

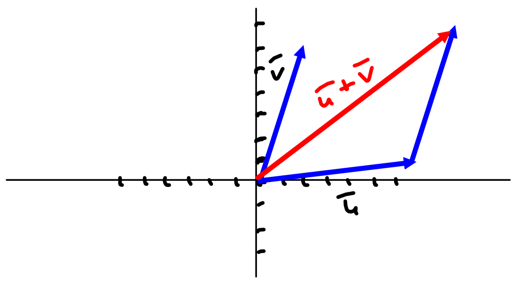
d. (5 pts) The resultant of $\overline{u}$ and $\overline{v}$.

$$\bar w = \bar u + \bar v = \langle 7,1\rangle + \langle 2,6\rangle = \boxed{\langle 9,7\rangle = \bar w}$$

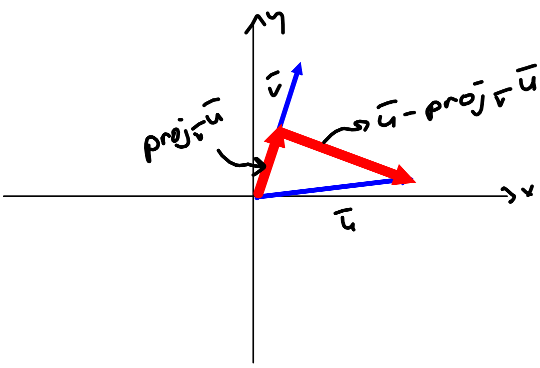
e. (5 pts) $\text{proj}_{\overline{v}}\left(\overline{u}\right)$. $= \dfrac{\overline{u}\cdot\overline{v}}{\|\overline{v}\|^2}\,\overline{v} = \dfrac{20}{(2^2+6^2)}\langle 2,6\rangle = \dfrac{20}{40}\langle 2,6\rangle$

$= \boxed{\langle 1,3\rangle = \text{proj}_{\overline{v}}\,\overline{u}}$

f. (5 pts) Sketch the vectors $\overline{u}$, $\overline{v}$, and $\overline{u}+\overline{v}$ on the same set of coordinate axes.



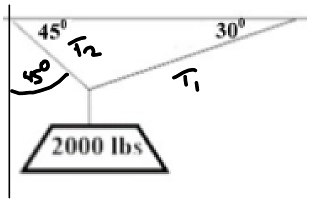
g. (5 pts) Sketch the vectors $\overline{u}$, $\overline{v}$, $\text{proj}_{\overline{v}}\left(\overline{u}\right)$, and $\overline{u}-\text{proj}_{\overline{v}}\left(\overline{u}\right)$ on the same set of coordinate axes. Don't over-write your sketch from part f. Start a new sketch,



6. (10 pts) A weight of 2000 lbs is suspended from two hooks in the ceiling. Find the tension in each cable. (See figure.) Round final answers to 3 decimal places.

$\frac{\pi}{12}x + \frac{3\pi}{2}$

$\frac{\pi}{12}(x-10) = \frac{\pi}{12}x - \frac{3\pi}{2}$

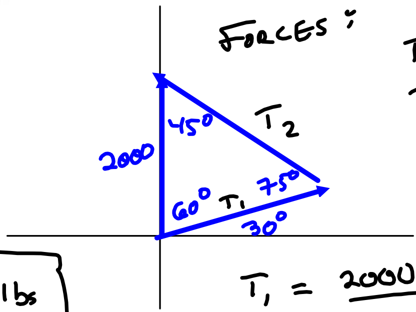


FORCES:

T_1 = tension in cable on right

T_2 = " " other cable

(see fig)



$\boxed{\begin{array}{l} T_1 \approx 1464.102 \text{ lbs} \\ T_2 \approx 1793.151 \text{ lbs} \end{array}}$

$$T_1 = \frac{2000 \sin(45^\circ)}{\sin(75^\circ)} \approx 1464.101615$$

$$T_2 = \frac{2000 \sin(60^\circ)}{\sin(75^\circ)} \approx 1793.150945$$

7. (10 pts) Use the fact that $z = 1 + 2i$ is a zero of

$$f(x) = x^4 - x^3 - 3x^2 + 17x - 30$$

to obtain a linear factorization of $f(x)$, that is, split f into linear factors by finding all its zeros.

Book Way

$(x-(1+2i))(x-(1-2i))$

$= x^2 - (1-2i)x - (1+2i)x + (1+2i)(1-2i)$

$= x^2 - x + 2ix - x - 2ix + 1^2 + 2^2$

$= x^2 - 2x + 5$

$$\begin{array}{r|l} & x^2 + x - 6 \\ \hline x^2-2x+5 & x^4 - x^3 - 3x^2 + 17x - 30 \\ & -(x^4 - 2x^3 + 5x^2) \\ \hline & x^3 - 8x^2 + 17x - 30 \\ & -(x^3 - 2x^2 + 5x) \\ \hline & -6x^2 + 12x - 30 \\ & -(-6x^2 + 12x - 30) \\ \hline & 0 \end{array}$$

Solve $x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3, 2$

Put it all together

$\boxed{(x-(1+2i))(x-(1-2i))(x-2)(x+3)}$

8. Let $z = 3\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ and $w = 7\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$. Compute the following.

a. (10 pts) zw

$$= \boxed{21\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right)}$$

Scratch

$$\frac{\pi}{3} + \frac{3\pi}{4} = \frac{4\pi + 9\pi}{12} = \frac{13\pi}{12}$$

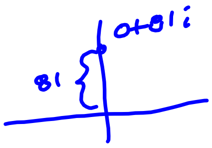
$$\frac{\pi}{3} - \frac{3\pi}{4} = \frac{4\pi - 9\pi}{12}$$

b. (10 pts) $\frac{z}{w}$

$$= \boxed{\frac{3}{7}\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)}$$

9. Let $z = 81i$.

a. (10 pts) Convert z to trigonometric form.



$|z| = 81$

$\theta = \frac{\pi}{2}$

$$z = 81\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

b. (10 pts) Find all 5^{th} roots of z. Leave your answers in trigonometric form. Use exact radians (i.e., π-radians).

$n = 5$, increment $= \frac{2\pi}{n} = \frac{2\pi}{5} = \frac{4\pi}{10}$

$k = 0$: $\frac{\theta}{n} = \frac{\theta}{5} = \frac{\frac{\pi}{2}}{5} = \frac{\pi}{10}$

$$\boxed{\sqrt[5]{81}\left(\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right)}$$

$k = 1$: $\sqrt[5]{81}\left(\cos\left(\frac{5\pi}{10}\right) + i\sin\left(\frac{5\pi}{10}\right)\right)$

$$= \boxed{\sqrt[5]{81}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)}$$

$k = 2$: $\boxed{\sqrt[5]{81}\left(\cos\left(\frac{9\pi}{10}\right) + i\sin\left(\frac{9\pi}{10}\right)\right)}$

$k = 3$: $\boxed{\sqrt[5]{81}\left(\cos\left(\frac{13\pi}{10}\right) + i\sin\left(\frac{13\pi}{10}\right)\right)}$

$k = 4$: $\boxed{\sqrt[5]{81}\left(\cos\left(\frac{17\pi}{10}\right) + i\sin\left(\frac{17\pi}{10}\right)\right)}$

$1 + 4 = 5$

$5 + 4 = 9$

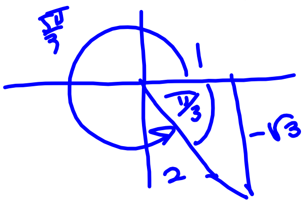
$9 + 4 = 13$

$13 + 4 = 17$

$17 + 4 = 21$

$\frac{21\pi}{10} = 2\pi + \frac{\pi}{10}$ ✓

c. (10 pts) Convert $z = 3\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$ to standard (rectangular) form.



$$z = 3\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \boxed{\frac{3}{2} - \frac{3\sqrt{3}}{2}i} \approx \boxed{1.5 - 2.5980762121}$$

10. Consider the following:

$x = 2 + 3\cos(\theta)$ $\Rightarrow x - 2 = 3\cos\theta \Rightarrow \frac{x-2}{3} = \cos\theta \Rightarrow \frac{(x-2)^2}{3^2} + \frac{(3-y)^2}{5^2} = 1$

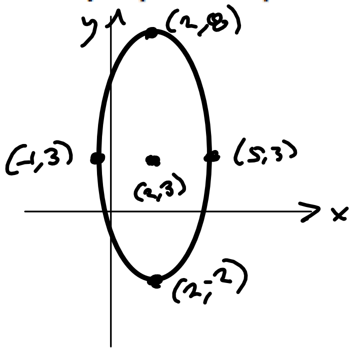
$y = 3 - 5\sin(\theta)$ $\Rightarrow y - 3 = -5\sin\theta \Rightarrow \frac{3-y}{5} = \sin\theta$

a. (10 pts) Sketch the curve represented by the parametric equations. Indicate the orientation of the curve.

$\frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{5^2} = 1$

$a = 3, b = 5$

$(h,k) = (2,3)$



b. (10 pts) Eliminate the parameter and write the resulting rectangular equation whose graph represents the curve.

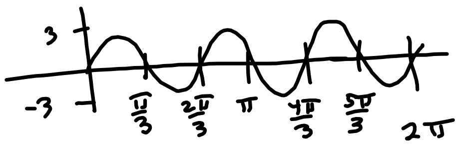
$$\boxed{\frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{5^2} = 1}$$

11. (10 pts) Convert the equation $2x - 3y - 15 = 0$ to polar form. Solve your equation for r so that it is of the form $r = f(\theta)$.

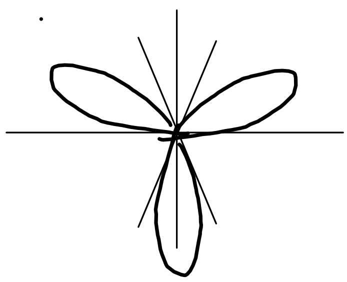
$2r\cos\theta - 3r\sin\theta = 15$

$$\boxed{r = \frac{15}{2\cos\theta - 3\sin\theta}}$$

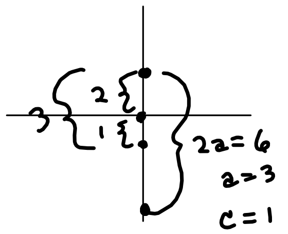
12. (10 pts) Sketch the graph of the polar equation $r = \sin(3\theta)$



A loop every $\frac{\pi}{3}$



13. (10 pts) Find the polar equation of the ellipse with vertices $\left(2,\frac{\pi}{2}\right),\left(4,\frac{3\pi}{2}\right)$.



$2a=6$

$a=3$

$c=1$

$e=\frac{c}{a}=\frac{1}{3}$

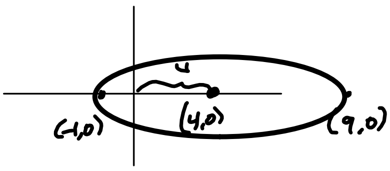
$r=\frac{ep}{1+e\sin\theta}=\frac{\frac{1}{3}p}{1+\frac{1}{3}\sin\theta}=\frac{p}{3+\sin\theta}$

$r\left(\frac{\pi}{2}\right)=\frac{p}{3+1}=2$

$p=8$

$r=\frac{\frac{1}{3}(8)}{1+\frac{1}{3}\sin\theta}=\boxed{\frac{8}{3+\sin\theta}=r}$

14. (Bonus 10 pts) Convert the equation $\frac{(x-4)^2}{25}+\frac{y^2}{9}=1$ to polar form of a conic $r=\frac{ep}{1-e\cos(\theta)}$.



$c^2=a^2-b^2=25-9=16$

$c=4$ ✓

$c=4$ $\quad e=\frac{c}{a}=\frac{4}{5}$

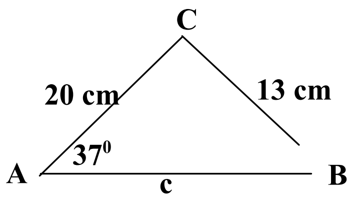
$a=5$

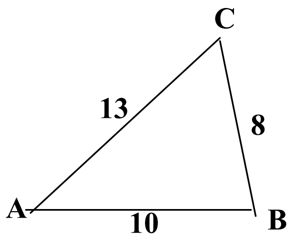
$r(0)=9=\frac{\frac{4}{5}p}{1-\frac{4}{5}}=\frac{4p}{5-4}=4p=9$

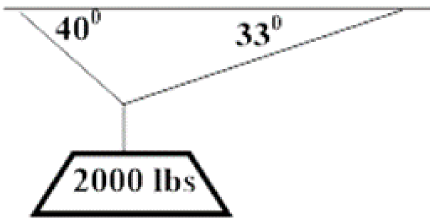
$\Rightarrow p=\frac{9}{4}$

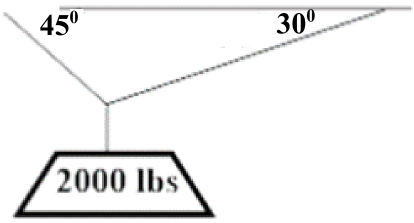
$r(\pi)=1=\frac{\frac{4}{5}(9/4)}{1-\frac{4}{5}(-1)}=\frac{\frac{9}{5}}{1+\frac{4}{5}}=\frac{\frac{9}{5}}{\frac{9}{5}}=1$ ✓

$r=\frac{\left(\frac{4}{5}\right)\left(\frac{9}{4}\right)}{1-\frac{4}{5}\cos\theta}=\frac{\frac{9}{5}}{1-\frac{4}{5}\cos\theta}=\boxed{\frac{9}{5-4\cos\theta}=r}$


C
20 cm
13 cm
37^0
A
c
B


C
13
8
A
10
B


40°
33°
2000 lbs


45°
30°
2000 lbs

$$2x-3y+1=0$$

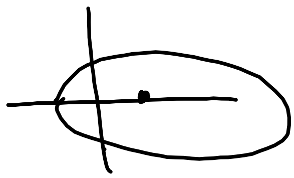
$$2r\cos\theta-3r\sin\theta+1=0$$

$$r(2\cos\theta-3\sin\theta)=-1$$

$$r=\frac{-1}{2\cos\theta-3\sin\theta}$$

$$\frac{(x-\cancel{2})^2}{9}+\frac{y^2}{25}=1 \qquad (h,k)=(2,0)$$

$$\frac{r^2\cos^2\theta}{9}+\frac{r^2\sin^2\theta}{25}=1$$

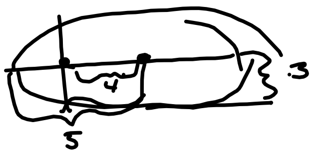


$$a^2=9,\ b^2=25$$

$$c^2=b^2-a^2=25-9=16$$

$$c=4$$

$$\frac{(x-4)^2}{25}+\frac{y^2}{9}-1$$



$$\frac{(x-4)^2}{25}+\frac{y^2}{9}=1 \qquad c^2=25-9=16$$

$$c=4$$

$$e=\frac{c}{a}=\frac{4}{5}$$

$$r=\frac{ep}{1-e\cos\theta}$$

$$\frac{ep}{1-e\cos\pi}=1$$

$$\frac{\frac{4}{5}p}{1-\frac{4}{5}(-1)}=\frac{4p}{5+4}=\frac{4p}{9}=1$$

$$4p=9$$

$$p=\frac{9}{4}$$

$$r=\frac{\frac{4}{5}\left(\frac{9}{4}\right)}{1-\frac{4}{5}\cos\theta}=\frac{9}{5-4\cos\theta}$$

$$r(0)=\frac{9}{5-4}=9$$