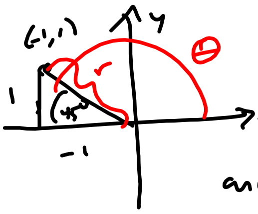
§6.7 - Polar Coordinates

$(x,y) \longmapsto (r,\theta)$



$x = r\cos\theta$ $\qquad$ $x^2+y^2=r^2$

$y = r\sin\theta$ $\qquad$ $\frac{y}{x} = \tan\theta$

$\arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{-1}\right) = -45^\circ = -\frac{\pi}{4}$

We're in QII, so

$\arctan\left(\frac{y}{x}\right) + \pi = \frac{3\pi}{4}$

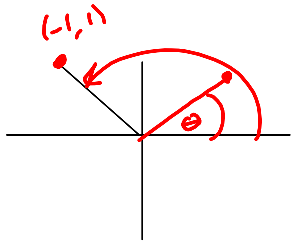
$\arctan\left(\frac{y}{x}\right) + 180^\circ = 135^\circ$

$r^2 = x^2+y^2 = 1^2+1^2 = 2 \Rightarrow r = \pm\sqrt{2}$

$(r,\theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$ is not unique!

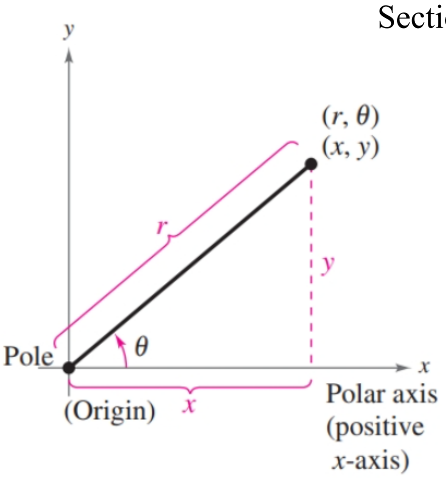
$= \left(-\sqrt{2}, -\frac{\pi}{4}\right)$ Negative length!

$= \left(\sqrt{2}, \frac{3\pi}{4} + 2\pi n\right)$ $n \in \mathbb{Z}$



oops! Not quite ready for symmetry!

Section 6.7 - Polar Coordinates



Coordinate Conversion

The polar coordinates (r, θ) and the rectangular coordinates (x, y) are related as follows.

Polar-to-Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Rectangular-to-Polar

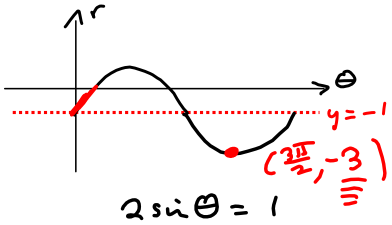
$$\tan\theta = \frac{y}{x}$$

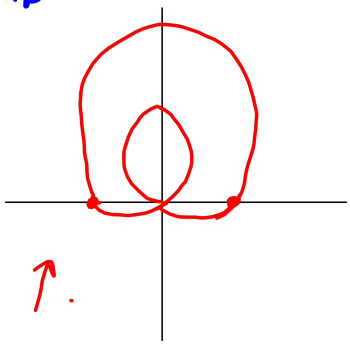
$$r^2 = x^2 + y^2$$

$r = 2\sin\theta - 1$

① Make rectangular graph.

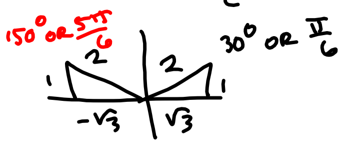
② " polar graph

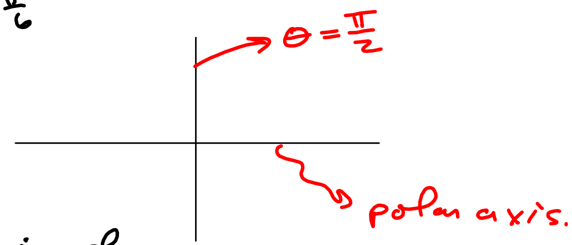




$2\sin\theta = 1$

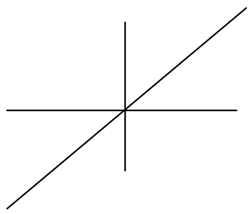
$\sin\theta = \frac{1}{2}$





Line thru the origin in polar coordinates:

$\theta = \text{constant}$, e.g., $\theta = \frac{\pi}{4}$



$\tan\frac{\pi}{4} = \frac{y}{x} = 1$

$\boxed{y = x}$ Rectangular.

Circle

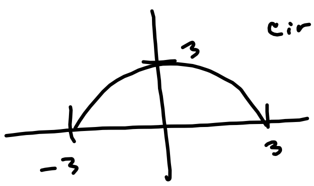
$x^2+y^2=4$

$r^2 = 4$

$r = \pm 2$

$r=2$ captures it all!

Polar coords are GREAT for circular objects.



circle of radius 3

rectangular

$x^2+y^2=3^2 \Longrightarrow y = \pm\sqrt{9-x^2}$

Top half. $= \sqrt{9-x^2}$ TOP HALF

Find the area.

$r = 3$

Area from $x=-3$ to $x=+3$ of $\sqrt{9-x^2}$ is hard in calculus.

In polars:

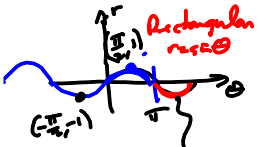
$$\left.\begin{aligned} 0 &\le r \le 3 \\ 0 &\le \theta \le \pi \end{aligned}\right\} \text{Noice!}$$

Graphing in Polar Coordinates.

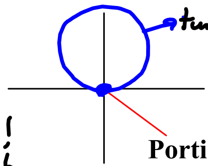
KEY: Graphing in Rectangular coordinates & seeing the "loops!"

$r = \sin\theta$ in polar coords.

First graph in rectangular coords



Polar



turns out it's a circle!

↳ r is negative! Flip through the pole! (origin)

Portions of the rectangular graph that are below the horizontal axis are their own little loops. For this one, it's the same loop. We go around the circle twice.

See Symmetry, S6.8 pg 2 (Pg 478 in 10th Ed.)

Symmetry about $\theta = \frac{\pi}{2}$ (Symmetry about y-axis)

$r = \sin\theta \qquad (r,\theta) \longmapsto (-r,-\theta)$

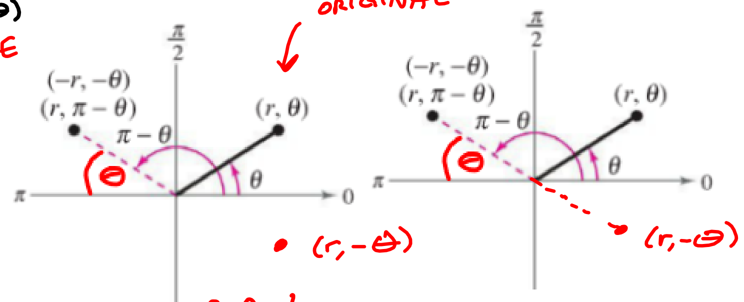
$-r = \sin(-\theta) = -\sin\theta$

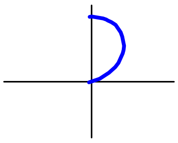
$\Rightarrow r = \sin\theta = \text{Same}!$ ← POSITIVE TEST

(Equivalent Equation!)

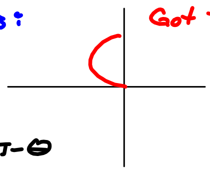
Everything from 0 to $\frac{\pi}{2}$ is mirrored in QII

ORIGINAL





gives us this:



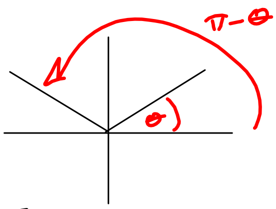
Got this for free!

Sometimes one test works and another doesn't. You just need it to work once.

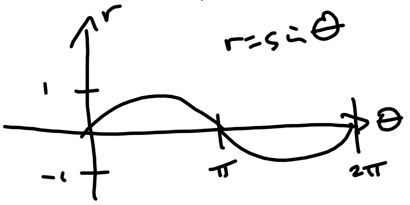
Alternate: Replace θ by $\pi - \theta$

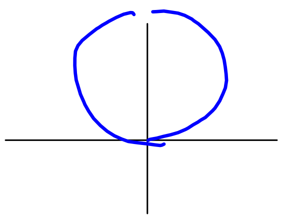
$r = \sin\theta \longmapsto r = \sin(\pi - \theta) = \sin\pi\cos(-\theta) + \sin(-\theta)\cos\pi$

$= 0 \cdot \underset{\text{even}}{\cos\theta} - (\sin\theta)(-1) = \sin\theta$! Same!



Mirrors across $\theta = \frac{\pi}{2}$ axis (which used to be y-axis)

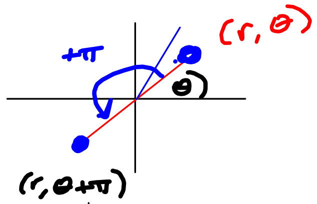


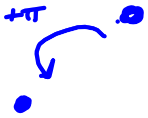


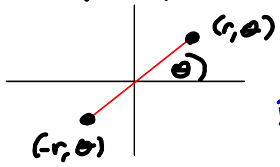
Symmetry

Symmetry through the pole:

$$\boxed{\begin{array}{l}(r,\theta) \longmapsto (r, \theta+\pi) \\ (r,\theta) \longmapsto (-r, \theta)\end{array}}$$







You see this kind of symmetry in something like

$r^2 = \sin\theta$

$(-r)^2 = \sin\theta$

$(-1)^2 r^2 = r^2 = \sin\theta$ Same!

Symmetry:

Pole: $r^2 = \sin\theta$

$r^2 = \sin(\theta+\pi)$

$= \sin\theta\cos\pi + \cancel{\sin\pi\cos\theta}$

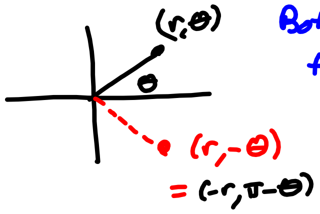
$r^2 = -\sin\theta$ NO

$(-r)^2 = \sin\theta$

$r^2 = \sin\theta$ = Same as original! So, symmetric thru origin.

3rd Kind of Symmetry:

Thru the polar axis (x-axis)

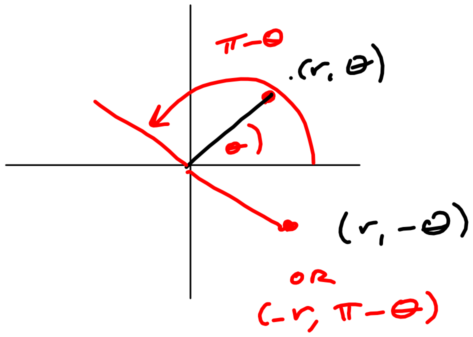


Bottom half of picture from top half.

$(r,\theta) \longmapsto (r,-\theta)$

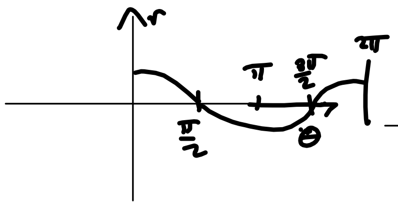
OR

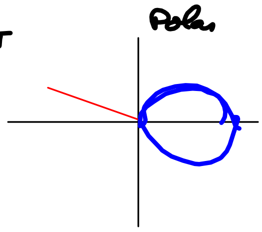
$(r,\theta) \longmapsto (-r,\pi-\theta)$



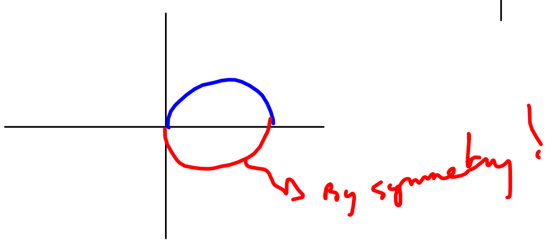
If one holds up, that's good enough.

$r=\cos\theta \rightsquigarrow r=\cos(-\theta)=\cos\theta$!





Circle!



$r=\cos\theta$

$r=\cos(-\theta)=\cos\theta$!

$r=\cos\theta$!

It's enough for ONE Test to "work"

$(-r,\pi-\theta)$:

$(-r)=\cos(\pi-\theta)$

$-r=\cos\pi\cos(-\theta)-\cancel{\sin\pi\sin(-\theta)}$

$-r=-1\cos\theta$

$r=\cos\theta$! Cool!

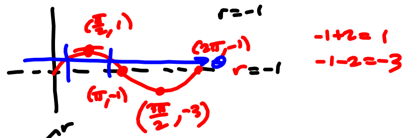
Section 6.9 is the last bit.

Amp = 2

$r = 2\sin\theta - 1$

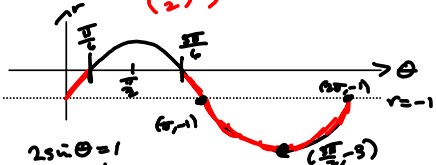
$\to y = -1$ midline

$r = -1$



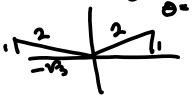
$-1 + 2 = 1$

$-1 - 2 = -3$

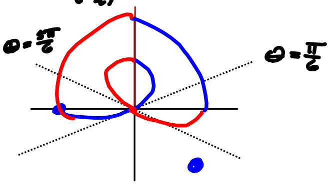


$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$



$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad \theta = \frac{5\pi}{6}$



$r = 2\sin\theta - 1$

$r = 2\sin(\pi - \theta) - 1$

$= 2(\sin\pi\cos(-\theta) + \sin(-\theta)\cos\pi) - 1$

$= 2\sin\theta - 1$! woo-hoo!

Symmetry thru $\theta = \frac{\pi}{2}$

$r = 2\sin(-\theta) - 1$ Polar Axis?

$= -2\sin\theta - 1$ not the same

$-r = 2\sin(\pi - \theta) - 1$

$-r = 2\sin\theta - 1$ not same.

Thru pole?

$r = 2\sin(\pi + \theta) - 1$

$= 2(\sin\pi\cos\theta + \sin\theta\underbrace{\cos\pi}_{-1}) - 1$

$= 2(-\sin\theta) - 1$ No

Try $(-r, \theta)$

$-r = 2\sin\theta - 1$ not same

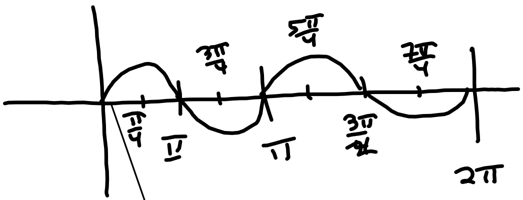
Wolfram Alpha Graph: I typed in **polar plot (2sin(x)-1) x=0 to x=2Pi**

If you go to Wolfram Alpha and type the same thing, you'll get this:

https://www.wolframalpha.com/input?i=polar+plot+%282sin%28x%29-1%29+x%3D0+to+2Pi

I barely worry about symmetry, until I'm close to done. Exception $r = \sin(2\theta)$, $r = \cos(3\theta)$, $r = \sin(3\theta)$ — flower petals

$r = \sin(2\theta)$



$\frac{\pi}{2}$-symmetry

$(r, \pi-\theta)$ OR $(-r, -\theta)$

$r = \sin(2(\pi-\theta))$

$= \sin(2\pi - 2\theta)$

$= \sin(2\pi)\cos(-2\theta) + \sin(-2\theta)\cos(2\pi)$

$= -\sin(2\theta)(1)$

$= -\sin(2\theta)$

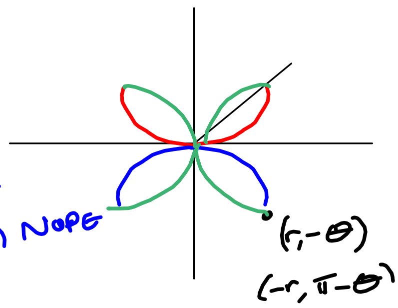
↑ NOT Same

$(-r, -\theta)$ test

$-r = \sin(-2\theta)$

$-r = -\sin(2\theta)$

$r = \sin(2\theta)$ Yes!



About polar axis:

$r = \sin(2(-\theta))$ NOPE

$-r = \sin(2(\pi-\theta))$

$-r = \sin(2\pi - 2\theta)$

$-r = \sin(-2\theta)$

$-r = -\sin(2\theta)$

$r = \sin(2\theta)$ Yes!

Symmetry about the pole (origin)

$(-r, \theta)$ OR $(r, \pi+\theta)$

$-r = \sin(2\theta)$? Nah.

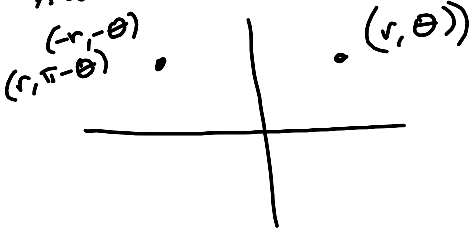
$r = \sin(2(\pi+\theta))$

$r = \sin(2\pi + 2\theta)$

$= \sin(2\theta)$ Yes!

Types of symmetry

About $\theta = \frac{\pi}{2}$

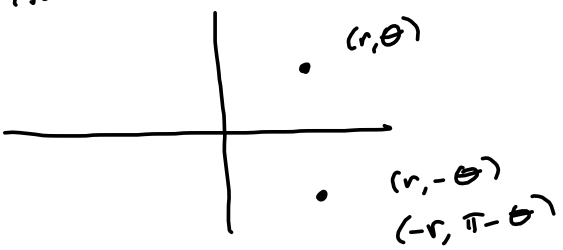


Check:

$-r = f(-\theta)$

$r = f(\pi - \theta)$

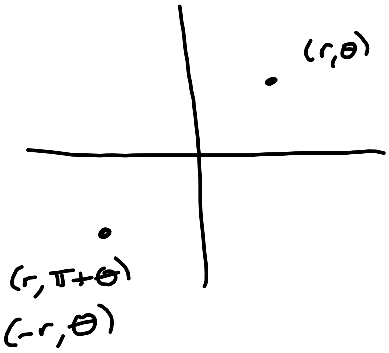
About Polar Axis



$r = f(-\theta)$

$-r = f(\pi - \theta)$

Through the pole:



$r = f(\theta + \pi)$

$-r = f(\theta)$

Next time:

Questions on 6.6 - 6.8

Begin 6.9