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HOR 103 between 7 am and 8 pm. Midterm over Chapters 1 & 2.

Give yourself 2 hours. One sided one page cheat sheet permitted.

Cheat Sheet Link

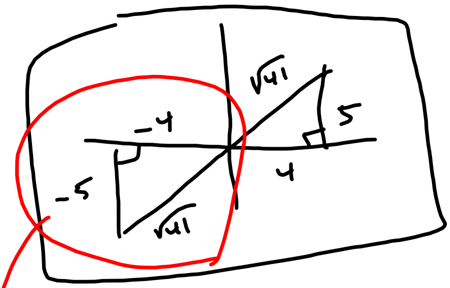
1. Arc Length and Area of Sector. Suppose we have a circle of radius $r = 10$.
   a. (5 pts) Find the arc length on the circle, that is intercepted by an angle of 1317^0. Round to 3 decimal places.
   b. (5 pts) Find the *exact* area of the sector that is intercepted (swept through) by an angle of $\theta = \frac{3\pi}{4}$

(a) $S = r\theta = (10)(1317^0)\left(\frac{\pi}{180^0}\right) = \frac{1317\pi}{18} = \boxed{\frac{439\pi}{6}}$ units?

(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2}(10^2)\left(\frac{3\pi}{4}\right) = \frac{1}{2}(25)(3\pi) = \boxed{150\pi}$ units^2

2. Answer the questions about the equation $\tan(\theta) = \frac{5}{4}$.

   a. (5 points) Sketch two triangles that satisfy $\tan(\theta) = \frac{5}{4}$.

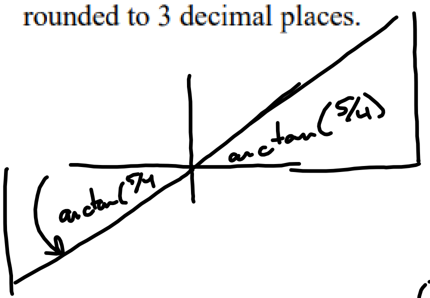


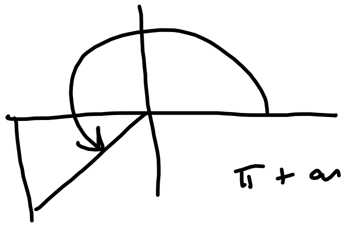
$4^2 + 5^2 = 16 + 25 = 41$

   b. (5 pts) Assume the terminal side of the angle θ lies in the 3rd quadrant. Find the other five trigonometric functions of θ.

$$\boxed{\begin{array}{ll} \sin\theta = -\frac{5}{\sqrt{41}} & \csc\theta = -\frac{\sqrt{41}}{5} \\ \cos\theta = \frac{-4}{\sqrt{41}} & \sec\theta = -\frac{\sqrt{41}}{4} \\ \tan\theta = \frac{5}{4} & \cot\theta = \frac{4}{5} \end{array}}$$

c. (5 pts) Again, assuming θ's terminal side lies in Q III, and $0 \le \theta < 2\pi$, find θ, in radians *and* degrees, rounded to 3 decimal places.





$\pi + \arctan\left(\frac{5}{4}\right)$

$4.037648038 \approx \boxed{4.038}$ OR $231.3401917^\circ \approx \boxed{231.340^\circ}$

$=0.8960553846 \approx \arctan(5/4)$

d. (5 pts) Give *all* solutions to the equation $\tan(\theta) = \frac{5}{4}$, in degrees *and* radians, rounded to three (3) decimal places.

$4.038 + 2n\pi$, $.896 + 2n\pi$

$231.340^\circ + 360^\circ n$, $51.340^\circ + 360^\circ n$

one way

$n \in \mathbb{Z}$

n is an integer.

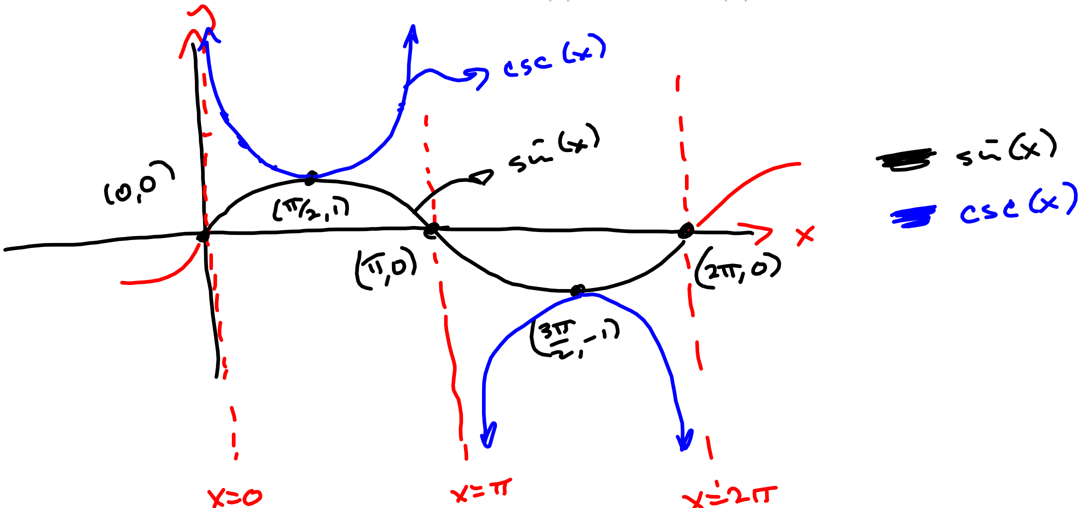
$.896 + n\pi$,

$51.340^\circ + 180^\circ n$, $n \in \mathbb{Z}$

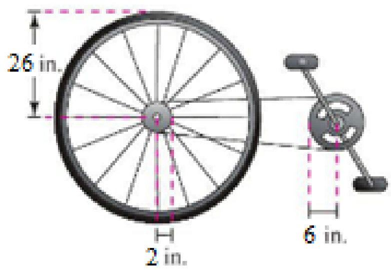
slick answer

$$\begin{array}{r} 231.340^\circ \\ -\ 180.000^\circ \\ \hline 51.340^\circ \end{array}$$

3. (5 pts) Sketch one period of the graphs of $y = \sin(x)$ and $y = \csc(x)$ on the same set of coordinate axes.



4. The radii of the pedal sprocket, the wheel sprocket, and the wheel of the bicycle in the figure are 6 inches, 2 inches and 26 inches, respectively. A cyclist is pedaling at a rate of 1.5 revolutions per second.



$$\underbrace{\left(1.4\ \frac{\text{revs front}}{\text{sec}}\right)\left(\frac{2\pi}{1\ \text{rev front}}\right)\left(\frac{1\ \text{rev front}}{3\ \text{revs rear}}\right)}_{\text{radians/sec rear}} \cdot (26\ \text{in})\left(\frac{1\ \text{ft}}{12\ \text{in}}\right)$$

$$= \left(\frac{14}{10}\right)\left(\frac{2\pi}{36}\right)(26) = \frac{(14\pi)(26)}{5(36)} = \frac{(14)(13)\pi}{5(18)}$$

$$= \frac{7(13)\pi}{5(9)} = \boxed{\frac{91\pi}{45}\ \frac{\text{ft}}{\text{s}}}$$

a. (5 pts) Find the speed of the bicycle in feet per second.

b. (5 pts) Convert your answer, above, to miles per hour. Round final answers to 1 decimal place.

$$\left(\frac{91\pi\ \text{ft}}{45\ \text{sec}}\right)\left(\frac{60\ \text{mph}}{88\ \frac{\text{ft}}{\text{sec}}}\right) = \text{etc.}$$

5. (5 pts) Sketch the graph of $f(x)=3\sin\left(\frac{\pi}{6}x+18\pi\right)+6$

$= 3\sin\left(\frac{\pi}{6}(x+108)\right)+6$

Amplitude

Period:

$\frac{\pi}{6}x = 2\pi$

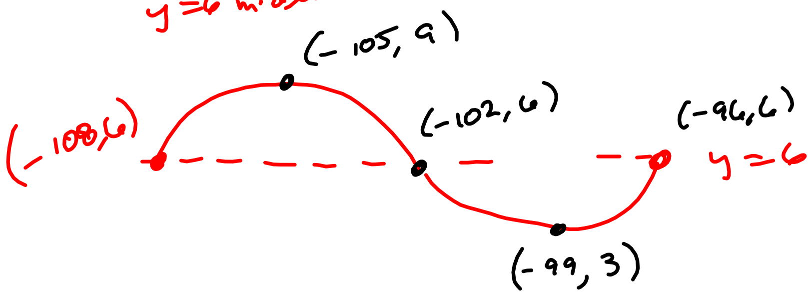
$x = \frac{(2\pi)(6)}{\pi} = \boxed{12 = \text{Period}}$

$\frac{\pi}{6}(x+108)$ start @ $x=-108$

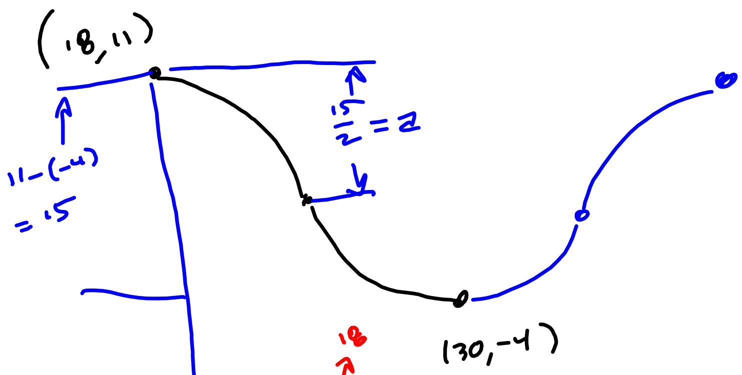
$\frac{18\pi}{\frac{\pi}{6}} =$

$\begin{array}{r} 18 \\ \times\ 6 \\ \hline 108 \end{array}$

y = 6 midline



6. (5 pts) Write the cosine function that achieves its maximum height of $y = 11$ centimeters at time $t = 18$ seconds and its minimum height of $y = -4$ centimeters at $t = 30$ seconds.



$30 - 12$ 18

$18 = \frac{1}{2}$ Period $12 = \frac{1}{2}$ Period!

Period $= 36$ $24 =$ Period!

Want

$bx = 2\pi$ when $x = 36$ 24

$36b = 2\pi$ 24

$b = \frac{2\pi}{36} = \frac{\pi}{18}$ 24 12

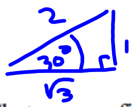
$2\cos\left(\frac{\pi}{18}(x - c)\right) + d$ 12

midline $\frac{-4 + 11}{2} = \frac{7}{2} = y$

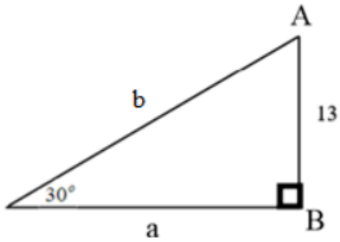
$2\cos\left(\frac{\pi}{18}(x - 18)\right) + \frac{7}{2}$ 12

Amplitude $= \frac{11 - (-4)}{2} = \frac{15}{2}$

$$\boxed{y = \frac{15}{2}\cos\left(\frac{\pi}{18}(x - 18)\right) + \frac{7}{2}}$$ 12



7. (5 pts) Solve the triangle in the figure on the right. That means, find all lengths and angles. Exact answers required.



$\frac{13}{b} = \sin 30^\circ = \frac{1}{2}$

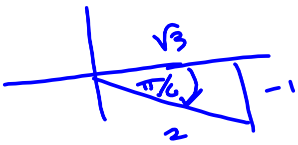
$\boxed{26 = b}$

$\frac{13}{a} = \tan 30^\circ = \frac{1}{\sqrt{3}}$

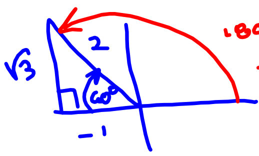
$(\sqrt{3})(13) = a = \boxed{13\sqrt{3} = a}$

$\boxed{A = 60^\circ,\ B = 90^\circ}$

8. (5 pts) Find the exact value of $\arccos\left(\sin\left(\frac{11\pi}{6}\right)\right) = \theta$



$= \arccos\left(-\frac{1}{2}\right) =$

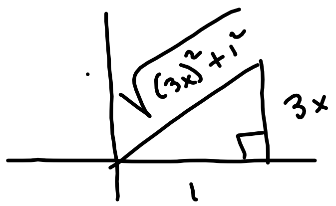


$180^\circ - 60^\circ = 120^\circ$

$\pi - \frac{\pi}{3} = \boxed{2\pi/3 = \theta}$

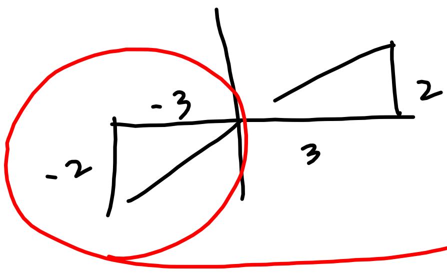
9. (5 pts) Draw the sketch and use it to find an algebraic expression that is equivalent to $\sin(\arctan(3x))$.

$\arctan(3x) = \theta$ $= \sin(\theta)$



$\Rightarrow$ $\boxed{\sin\theta = \dfrac{3x}{\sqrt{9x^2+1}}}$

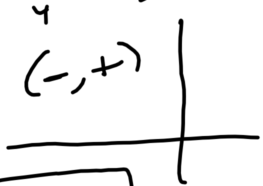
10. (5 pts) Find the values of all six trigonometric functions, given $\tan(u) = \dfrac{2}{3}$ and $\sin(u) < 0$.



11. (5 pts) Find $\sin\left(\frac{u}{2}\right)$, $\cos\left(\frac{u}{2}\right)$, and $\tan\left(\frac{u}{2}\right)$, given that $\cos(u) = \frac{3}{4}$ and $\frac{3\pi}{2} < u < 2\pi$

$\frac{3\pi}{2} < u < 2\pi \Rightarrow$

$\frac{3\pi}{4} < \frac{u}{2} < \pi \Rightarrow$ Q II $\Rightarrow$ cosine's neg & sine's pos.



$\sin(\frac{u}{2}) = -\sqrt{\frac{1-\cos u}{2}} = +\sqrt{\frac{1-\frac{3}{4}}{2}}$

$+$

$= +\sqrt{\frac{\frac{1}{4}}{2}} = +\sqrt{\frac{1}{2\cdot 4}} = \boxed{+\frac{1}{2}\sqrt{\frac{1}{2}}}$

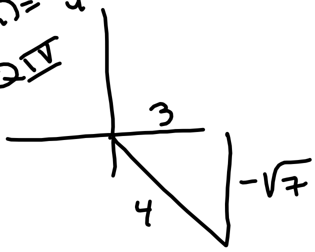
$\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}$

$\boxed{\sin(\frac{u}{2}) = +\frac{1}{2}\sqrt{\frac{1}{2}}} = -\frac{\sqrt{2}}{4}$

$\cos(\frac{u}{2}) = -\sqrt{\frac{1+\cos(u)}{2}} = -\sqrt{\frac{\frac{7}{4}}{2}} = -\sqrt{\frac{7}{2\cdot 4}} = \boxed{-\frac{1}{2}\sqrt{\frac{7}{2}} = \cos(\frac{u}{2})}$

$\tan(\frac{u}{2}) = \frac{1-\cos(u)}{\sin(u)} = \frac{1-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{\frac{1}{4}}{-\frac{\sqrt{7}}{4}} = -\frac{1}{4}\cdot\frac{4}{\sqrt{7}} = \boxed{-\frac{1}{\sqrt{7}} = \tan(\frac{u}{2})}$

$\cos(u) = \frac{3}{4}$ & $u \in$ Q IV



$4^2 - 3^2 = 16 - 9 = 7$

Another way: $\tan(\frac{u}{2}) = \frac{\sin(\frac{u}{2})}{\cos(\frac{u}{2})} = \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{-\frac{1}{2}\sqrt{\frac{7}{2}}} = -\frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{7}}{\sqrt{2}}}$

$= \boxed{-\frac{1}{\sqrt{7}} = \tan(\frac{u}{2})}$

12. Consider the equation $2\sin^2(x) - 1 = 0$.

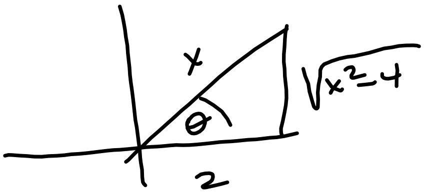
a. (5 pts) Find all solutions x, in radians *and* degrees, to the equation in the interval $[0, 2\pi)$.

b. (5 pts) Find all real solutions x, in radians *and* degrees.

13. (5 pts) Re-write $\sin\left(\cos^{-1}\left(\frac{2}{x}\right)\right)$ as an algebraic expression.

$= \sin\theta$

$\cos^{-1}\left(\frac{2}{x}\right) = \arccos\left(\frac{2}{x}\right)$



$$\boxed{\sin\theta = \frac{\sqrt{x^2-4}}{x}}$$

14. Find the *exact* value of $\sin\left(\frac{5\pi}{6}\right)$ in two ways: (Hint: If degrees are easier for you, *use degrees*.) → No.

a. (5 pts) Use a Sum identity.
b. (5 pts) Use a Half-Angle identity

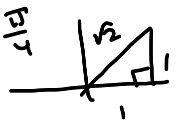
$\frac{5\pi}{6} = \frac{2\pi}{6} + \frac{3\pi}{6} = \frac{\pi}{3} + \frac{\pi}{2}$

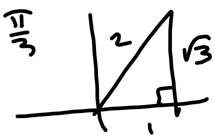
$\sin\left(\frac{7\pi}{12}\right)$

$7 = 1+6$
$= 2+5$
$= 3+4$

$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$

$\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) =$





$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$

$= \boxed{\frac{1+\sqrt{3}}{2\sqrt{2}}}$

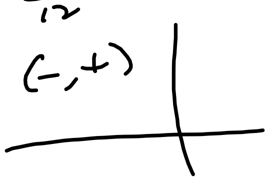
$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\frac{7\pi}{6}}{2}\right)$

$u = \frac{7\pi}{6}$

$\sin\frac{u}{2} = + \sqrt{\frac{1-\cos(u)}{2}}$ etc.

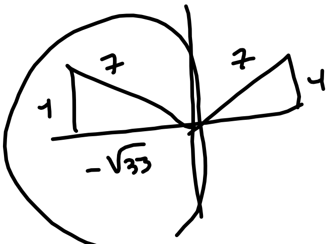
$\frac{7\pi}{12} \in QII$

$(-,+)$



Bonus on midterm.

16. (5 pts) Find $\sin(2u), \cos(2u)$ and $\tan(2u)$, given that $\sin(u) = \frac{4}{7}$ and $\cos(u) < 0$.



$49 - 16 = 33$

$$\sin(2u) = 2\sin(u)\cos(u) = 2\left(\frac{4}{7}\right)\left(-\frac{\sqrt{33}}{7}\right) = \boxed{-\frac{8\sqrt{33}}{49} = \sin(2u)}$$

$$\cos(2u) = 2\cos^2(u) - 1 = 2\left(-\frac{\sqrt{33}}{7}\right)^2 - 1$$

$$= \frac{2\cdot 33}{49} - \frac{49}{49} = \frac{66-49}{49} = \boxed{\frac{17}{49} = \cos(2u)}$$

$$\tan(2u) = \frac{\sin(2u)}{\cos(2u)} = \left(-\frac{8\sqrt{33}}{49}\right)\left(\frac{49}{17}\right) = \boxed{-\frac{8\sqrt{33}}{17} = \tan(2u)}$$

$$= \frac{2\tan(u)}{1-\tan^2(u)} = \frac{2\left(-\frac{4}{\sqrt{33}}\right)}{1-\left(\frac{16}{33}\right)} = \frac{-\frac{8}{\sqrt{33}}}{\frac{33-16}{33}} = \frac{-\frac{8}{\sqrt{33}}}{\frac{17}{33}}$$

$$= \frac{-8}{\sqrt{33}} \cdot \frac{33}{17} = \frac{(-8)(33)}{17\sqrt{33}} =$$

$$= \frac{-8(33)}{(17\sqrt{33})}\left(\frac{\sqrt{33}}{\sqrt{33}}\right)$$

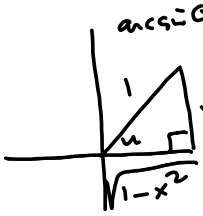
$$= \frac{-8(33)\sqrt{33}}{17(33)} = -\frac{8\sqrt{33}}{17}$$

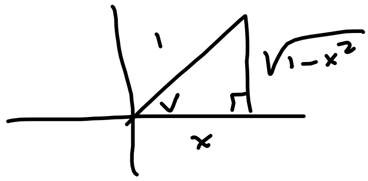
15. (5 pts) Re-write $\cos\left(\arcsin(x)+\arccos(x)\right)$ as an algebraic expression. (Hint: Use Sum identity.)

$= \cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$

$= \cos(\arcsin(x))\cos(\arccos(x)) - \sin(\arcsin(x))(\sin(\arccos(x)))$

arcsin(x) arccos (x)





$= \frac{\sqrt{1-x^2}}{1} \cdot x - x \cdot \frac{\sqrt{1-x^2}}{1} = 0 \,!$