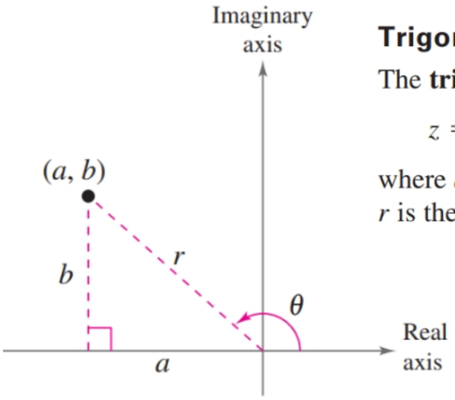


Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

Not "the" argument

$\theta + 2n\pi$ for any $n \in \mathbb{Z}$ would work.

$z_1 = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 1 + \sqrt{3}\, i$

$z_2 = 4\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = -2\sqrt{2} + 2\sqrt{2}\, i$

$z_1 z_2 = (1+\sqrt{3}\, i)(-2\sqrt{2} + 2\sqrt{2}\, i)$

$= -2\sqrt{2} + 2\sqrt{2}\, i - 2\sqrt{6}\, i + 2\sqrt{6}\, i^2$

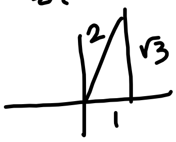
$= -2\sqrt{2} - 2\sqrt{6} + (2\sqrt{2} + 2\sqrt{6})\, i$

In Rectangular Form

$= (2\sqrt{2} - 2\sqrt{6}) + (2\sqrt{2} - 2\sqrt{6})\, i$

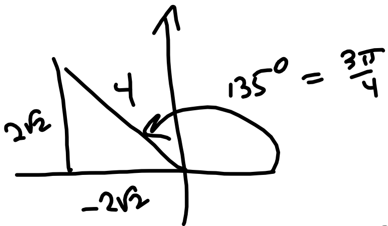
Trigonometric Form:

$z_1 = 2(\cos 60^\circ + i\sin 60^\circ) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$



z_2

$$\sqrt{(2\sqrt{2})^2+(2\sqrt{2})^2} = \sqrt{4\cdot 2 + 4\cdot 2} = \sqrt{(8+8)} = \sqrt{16} = 4$$



$$z_2 = 4\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

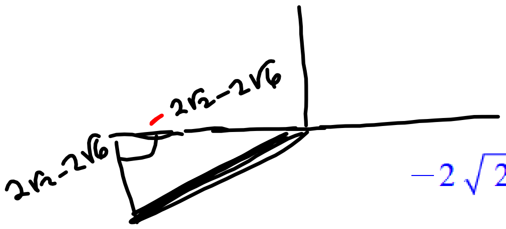
Now, write $z = z_1 z_2$ in trig form:

$$= (2\sqrt{2} - 2\sqrt{6}) + (2\sqrt{2} - 2\sqrt{6})i$$

$-$

$$= 2(-\sqrt{2} - \sqrt{6}) + 2(\sqrt{2} - 2\sqrt{6})i$$

Something hinky:



$$-2\sqrt{2} + 2\,I\sqrt{2} - 2\,I\sqrt{3}\,\sqrt{2} - 2\sqrt{3}\,\sqrt{2}$$

$$= -2\sqrt{2} - 2\sqrt{6}$$

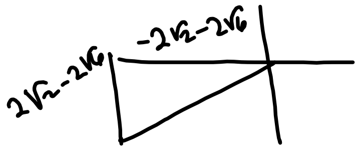
Length:

$$\sqrt{(-2\sqrt{2}-2\sqrt{6})^2 + (2\sqrt{2}-2\sqrt{6})^2}$$

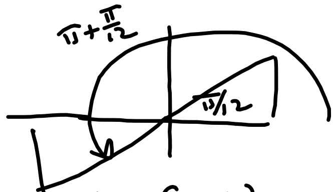
$$= \sqrt{4\cdot 2 + 2(2\sqrt{2})(2\sqrt{6}) + 4\cdot 6 + 4\cdot 2 - 2(2\sqrt{2})(2\sqrt{6}) + 4\cdot 6}$$

$$= \sqrt{8 + 8\sqrt{12} + 24 + 8 - 8\sqrt{12} + 24} = \sqrt{16+48} = \sqrt{64} = \boxed{8 = r!}$$

Now find θ!



$$\tan\theta = \frac{2\sqrt{2}-2\sqrt{6}}{-2\sqrt{2}-2\sqrt{6}} \Rightarrow$$



$$\arctan(\tan\theta) = \arctan(\text{mess}) = \frac{\pi}{12}$$

So our θ is $\pi + \frac{\pi}{12} = \frac{13\pi}{12}$

$$z = z_1 z_2 = 8\left(\cos\left(\tfrac{7\pi}{12}\right) + i\sin\left(\tfrac{7\pi}{12}\right)\right)$$

$r_1 = 2,\ r_2 = 4$

$r_1 r_2 = 8 = |z| = |z_1 z_2|$

$$\frac{3\pi}{4} + \frac{\pi}{3} = \frac{9\pi + 4\pi}{12} = \frac{13\pi}{12}$$

$$z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

This is the formula for multiplying two complex numbers in trigonometric form:

Multiply the moduli and add the angles (arguments)

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

DeMoivre's Theorem for Powers:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

is an "obvious" consequence for how we multiply two complex numbers in trigonometric form.

Every number has 2 square roots:

$x^2 = z \Rightarrow$

$x = \pm\sqrt{z}$

Any polynomial of degree n has n complex roots

Any power function of degree n has n complex n^{th} roots

$x^3 = 8$

$\sqrt[3]{x} = \sqrt[3]{8} = 2$, so

$x^3 - 8 = (x-2)$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$ sweet!

$(x-2)(x^2+2x+4)$

$b^2 - 4ac = 2^2 - 4(1)(4) = -12$

$$\begin{array}{r|l} 2 & 12 \\ 2 & 6 \\ & 3 \end{array}$$

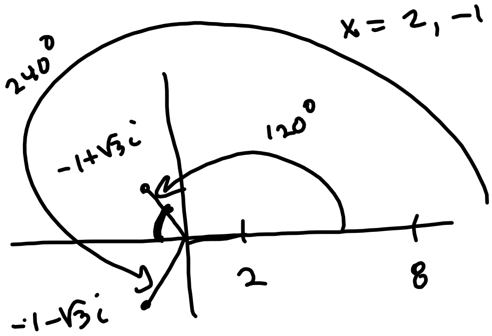
$2\sqrt{3}$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-2 \pm 2\sqrt{3}\,i}{2(1)} = -1 \pm \sqrt{3}\,i$

$\Rightarrow x^3 - 8 = (x-2)(x-(-1+\sqrt{3}\,i))(x-(-1-\sqrt{3}\,i))$

So $x^3 = 8$, when
$x = 2, -1 \pm \sqrt{3}\,i$



$2(\cos(0) + i\sin 0)$

$2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

They're all $\frac{2\pi}{3}$ apart.

Their lengths are all $\sqrt[3]{8} = 2$

4^{th} roots

Separated by angle of $\frac{2\pi}{4} = \frac{\pi}{2}$

$x^4 = 16$

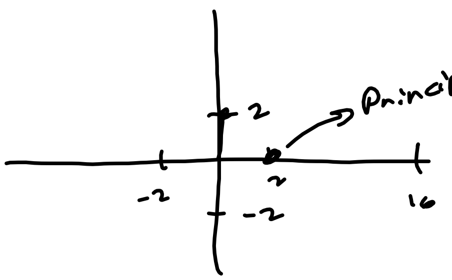
$2(\cos(0) + i\sin(0))$

$2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$2(\cos\pi + i\sin\pi)$ Same

$2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$

$2(\cos(2\pi) + i\sin(2\pi))$



This works for ANY Complex #s *n*th roots.

Find all the 5^{th} roots of $z = 32(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$\sqrt[5]{z} = 32^{\frac{1}{5}}(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20})$ Divide θ by 5

$= 2(\cos(\frac{\pi}{20}) + i\sin(\frac{\pi}{20}))$

To get the rest, just add the increment

$\frac{2\pi}{5} = \frac{2\pi}{n}$

What?

$$\frac{\pi}{20}+\frac{2\pi}{5}\cdot\frac{4}{4}=\frac{\pi}{20}+\frac{8\pi}{20}=\frac{28\pi}{20}=\frac{7\pi}{5}$$

$$\cancel{\frac{28\pi}{20}+\frac{8\pi}{20}=\frac{36\pi}{20}=\frac{9\pi}{5}} \qquad \cancel{\frac{9\pi}{5}+\frac{2\pi}{5}}$$

$$\cancel{\frac{36\pi}{20}+\frac{8\pi}{20}=\frac{44\pi}{20}=\frac{11\pi}{5}}$$

$$\cancel{\frac{44\pi}{20}+\frac{8\pi}{20}=\frac{52\pi}{20}=\frac{13\pi}{5}}$$

$$\cancel{\frac{52\pi}{20}+\frac{8\pi}{20}=\frac{60\pi}{20}=3\pi}$$

$2\left(\cos\left(\frac{\pi}{20}\right)+i\sin\left(\frac{\pi}{20}\right)\right)$ $k=0$ $\qquad \frac{\pi}{20}+\frac{2\pi}{5}\cdot\frac{4}{4}=\frac{9\pi}{20}$

$2\left(\cos\left(\frac{9\pi}{20}\right)+i\sin\left(\frac{9\pi}{20}\right)\right)$ $k=1$ $\qquad \frac{9\pi+8\pi}{20}=\frac{17\pi}{20}$

$2\left(\cos\left(\frac{17\pi}{20}\right)+i\sin\left(\frac{17\pi}{20}\right)\right)$ $k=2$ $\qquad \frac{(17+8)\pi}{20}=\frac{25\pi}{20},\ldots$

$k=3$ $\quad 2\left(\cos\left(\frac{25\pi}{20}\right)+i\sin\left(\frac{25\pi}{20}\right)\right)=2\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right)$

$k=4$ $\quad 2\left(\cos\left(\frac{33\pi}{20}\right)+i\sin\left(\frac{33\pi}{20}\right)\right)$

$k=5$ $\quad 2\left(\cos\left(\frac{41\pi}{20}\right)+i\sin\left(\frac{41\pi}{20}\right)\right)$ You've gone too far!

"mod out" the 2π & you're back

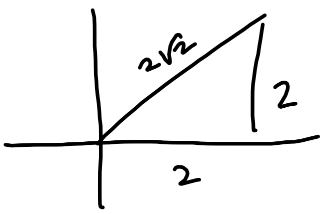
$\sqrt[5]{z}$

$$\frac{41\pi}{20}=\frac{40\pi}{20}+\frac{1\pi}{20}=2\pi+\frac{\pi}{20}, \ \&$$

we've come full circle!

$x^5=$ Anything has 5 solutions & we can generate them!

$(2+2i)^6 \qquad z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$



$z^6 = (2\sqrt{2})^6\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$

$\frac{6\pi}{4} = \frac{3\pi}{2}$

$= 2^6 (2^{\frac{1}{2}})^6 (\cos \sim\sim)$

$= 2^5\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$

$= 32\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$

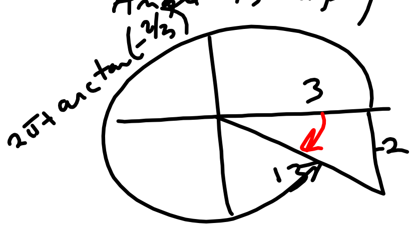
$\sqrt{2^2+2^2} = \sqrt{2\cdot 2^2}$

$= 2\sqrt{2}$

$(2\sqrt{2})^6 = (2\cdot 2^{\frac{1}{2}})^6 = (2^{\frac{3}{2}})^6 = 2^9$

$(3-2i)^6$?

Angle is ugly



$\sqrt{9+4} = \sqrt{13}$

$z = \sqrt{13}\left(\cos\left(\arctan\left(-\frac{2}{3}\right)+2\pi\right) + i\sin\left(\arctan\left(-\frac{2}{3}\right)+2\pi\right)\right)$

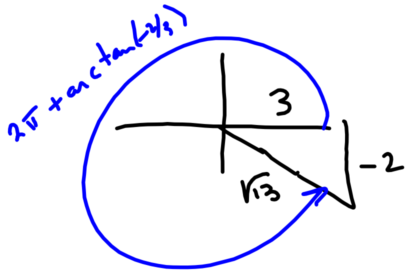
$z^6 = \left(13^{\frac{1}{2}}\right)^6\left(\cos\left(6\left(\arctan\left(-\frac{2}{3}\right)+2\pi\right)\right)\right.$

$\left.+\, i\sin\left(6\left(\arctan\left(-\frac{2}{3}\right)+2\pi\right)\right)\right)$

13^3

?

$\cos\left(\arctan\left(-\frac{2}{3}\right)\right) = \frac{3}{\sqrt{13}}$



$\sin\left(\arctan\left(-\frac{2}{3}\right)\right) = -\frac{2}{\sqrt{13}}$

$(\sqrt{13})^6\left(\cos\left(6\left(\arctan\left(-\frac{2}{3}\right)\right)\right)\right.$

$\left.+\, i\sin\left(6\left(\arctan\left(-\frac{2}{3}\right)\right)\right)\right)$

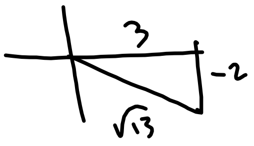
$$\begin{array}{r} 169 \\ \times\ 13 \\ \hline 507 \\ 1690 \\ \hline 2197 \end{array}$$

$13^3 = 2197$

$\cos\left(6\left(\arctan\left(-\frac{2}{3}\right)\right)\right)$

$= \cos(6)\cos\left(\arctan\left(-\frac{2}{3}\right)\right)$

$+ i\sin(6)\sin\left(\arctan\left(-\frac{2}{3}\right)\right)$



$= \cos(6)\cdot\frac{3}{\sqrt{13}} + i\sin(6)\left(\frac{-2}{\sqrt{13}}\right)$

One last try:

$$z^6 = (3-2i)^6$$

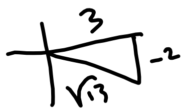
$$|z| = \sqrt{13}$$

$$\arg(z) = \arctan\left(-\tfrac{2}{3}\right)$$

$$z^6 = 13^3\left(\cos\left(6\arctan\left(-\tfrac{2}{3}\right)\right) + i\sin\left(6\arctan\left(-\tfrac{2}{3}\right)\right)\right)$$

$$= 13^3\Big(\cos(6)\cos\left(\arctan\left(-\tfrac{2}{3}\right)\right) - \sin(6)\sin\left(\arctan\left(-\tfrac{2}{3}\right)\right)$$

$$+ i\left(\sin(6)\cos\left(\arctan\left(-\tfrac{2}{3}\right)\right) + \sin\left(\arctan\left(-\tfrac{2}{3}\right)\right)\cos(6)\right)\Big)$$



$$= 13^3\left(\cos(6)\cdot\tfrac{3}{\sqrt{13}} - \sin(6)\left(-\tfrac{2}{\sqrt{13}}\right)\right.$$

$$\left. + i\left(\sin(6)\cdot\tfrac{3}{\sqrt{13}} + \cos(6)\cdot\left(-\tfrac{2}{\sqrt{13}}\right)\right)\right)$$

Holy Moley!

There's a "Watch It" video accompanying this exercise.

He skips right over the hard part. What a jerk!

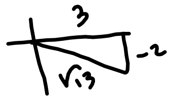
$$z^6 = 13^3\left(\cos\left(6\arctan\left(-\tfrac{2}{3}\right)\right) + i\sin\left(6\arctan\left(-\tfrac{2}{3}\right)\right)\right)$$

$$= 13^3\left(\cos\left(6\cdot\arctan\left(-\tfrac{2}{3}\right)\right) + i\sin\left(6\arctan\left(-\tfrac{2}{3}\right)\right)\right)$$

$$= 13^3\left(\cos(uv) + i\sin(uv)\right)$$

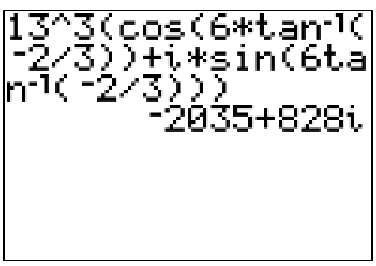
$$= 13^3\left(\cos u\cos v - \sin u\sin v + i\left(\sin u\cos v + \sin v\cos u\right)\right)$$

$$= 13^3\Big(\cos(6)\cos\left(\arctan\left(-\tfrac{2}{3}\right)\right) - \sin(6)\sin\left(\arctan\left(-\tfrac{2}{3}\right)\right)$$



$$+ i\left(\sin(6)\cos\left(\arctan\left(-\tfrac{2}{3}\right)\right) + \sin\left(\arctan\left(-\tfrac{2}{3}\right)\right)\cos(6)\right)\Big)$$

$$= 13^3\left(\cos(6)\cdot\tfrac{3}{\sqrt{13}} - \sin(6)\cdot\tfrac{-2}{\sqrt{13}} + i\left(\sin(6)\cdot\tfrac{3}{\sqrt{13}} + \cos(6)\cdot\tfrac{-2}{\sqrt{13}}\right)\right)$$



So I can get it, but it's all black box, using calculator to simplify that ungodly mess.