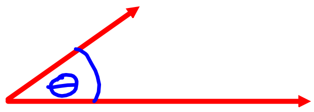
Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

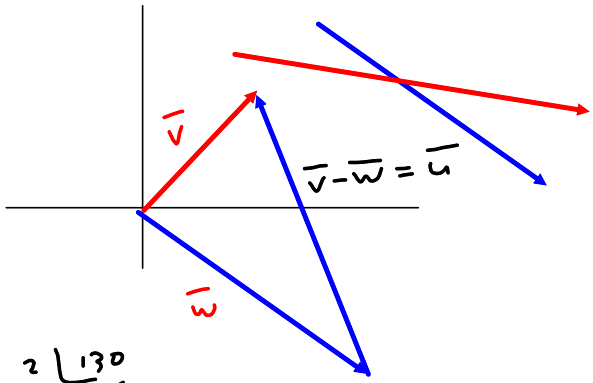


3.3 #34

$\bar{v} = 4\bar{i} + 4\bar{j} = \langle 4, 4 \rangle$

$\bar{w} - 7\bar{i} - 7\bar{j} = \langle 7, -7 \rangle$

$\bar{v} =$



$\| 4\langle 1, 1 \rangle \| = 4\sqrt{2}$

$\| 7\langle 1, -1 \rangle \| = 7\sqrt{2}$

$2 \,|\, 130$
$5 \,|\, 65$
13

$\bar{u} = \bar{v} - \bar{w} = \langle -3, 11 \rangle$

$\| \bar{u} \| = \sqrt{3^2 + 11^2} = \sqrt{130}$

This makes SSS for Law of Cosines.

You can do this, now.

ORTHOGONAL means $\perp$ (PERPENDICULAR)

$\cos(90^\circ) = 0$

DEF'N DOT PRODUCT:

$$\bar{v} \cdot \bar{w} = \langle 4,4 \rangle \cdot \langle 7,-7 \rangle$$

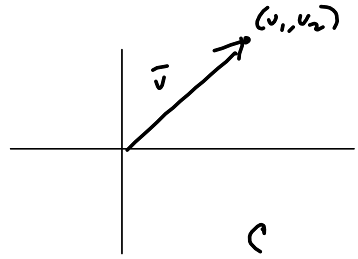
$$= (4)(7) + (4)(-7) = 28 - 28 = 0$$

FACT: $\bar{v} \cdot \bar{w} = 0$ if and only if $\bar{v} \perp \bar{w}$.

FACTS:

Length/Modulus/Magnitude/norm of

$\bar{v} = \sqrt{v_1^2 + v_2^2}$

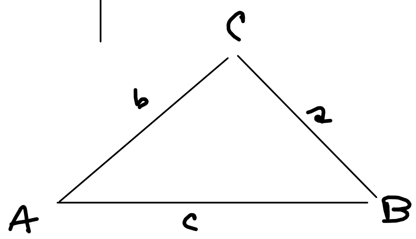


$$\bar{v} \cdot \bar{v} = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle$$

$$= v_1^2 + v_2^2 \; !$$

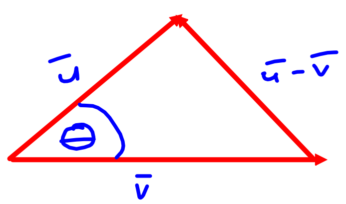
$$= \|\bar{v}\|^2$$

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$



In the language of vectors, we have:

$$\|\bar{u} - \bar{v}\|^2 = \|u\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos\theta$$

FACT $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

$$\|\bar{u} - \bar{v}\|^2 = (\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) = \bar{u} \cdot \bar{u} - \bar{u} \cdot \bar{v} - \bar{v} \cdot \bar{u} + \bar{v} \cdot \bar{v}$$

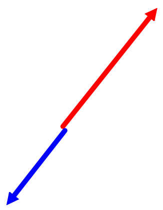
$$= \|\bar{u}\|^2 - 2\,\bar{u} \cdot \bar{v} + \|\bar{v}\|^2$$

$$= \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\,\|\bar{v}\| \cos\theta$$

$$\Rightarrow -2\|\bar{u}\|\,\|\bar{v}\| \cos\theta = -2\,\bar{u} \cdot \bar{v} \Rightarrow$$

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\,\|\bar{v}\|}$$

These angles never exceed 180 degrees



So $\cos^{-1}$(mess) always gives us the angle, with no further interpretation.

The angle between $\bar{v}$ & $\bar{w}$, using 3.4:

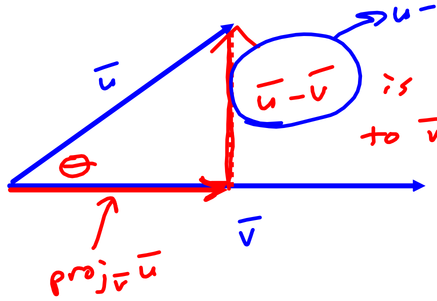
$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\,\|\bar{v}\|} = \frac{\bar{v} \cdot \bar{w}}{\|\bar{v}\|\,\|\bar{w}\|} = \frac{\langle 4,4\rangle \cdot \langle 7,-7\rangle}{(4^2+4^2)(7^2+(-7)^2)}$$

$$= \frac{28-28}{(32)(98)} = \frac{0}{(32)(98)} = 0 \Rightarrow$$

$$\theta = \cos^{-1}(0) = 90^\circ = \frac{\pi}{2} \; !$$

Projection of vectors...

$\text{proj}_{\bar{v}}\,\bar{u}$



$\to \bar{u} - \text{proj}_{\bar{v}}\,\bar{u}$!

is perpendicular to $\bar{v}$!

$$\cos\theta = \frac{\|\text{proj}_{\bar{v}}\,\bar{u}\|}{\|\bar{u}\|}$$

$$\|\bar{u}\|\cos\theta = \|\text{proj}_{\bar{v}}\,\bar{u}\| \longrightarrow \|\bar{u}\|\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|}$$

& p

Projection of u onto v

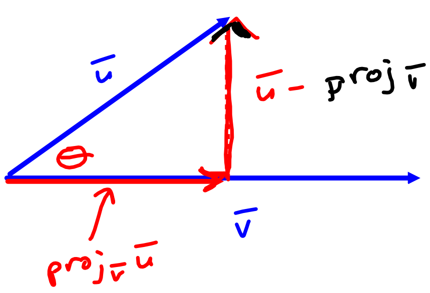
Let **u** and **v** be nonzero vectors. The projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

The idea is that $\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|}$ = Length of $\text{proj}_{\bar{v}}\,\bar{u} = \|\text{proj}_{\bar{v}}\,\bar{u}\|$

& $\frac{\bar{v}}{\|\bar{v}\|} = \frac{1}{\|\bar{v}\|}\bar{v}$ = a unit vector in the direction of $\bar{v}$.

$$\text{proj}_{\bar{v}}\,\bar{u} = \underbrace{\left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|}\right)\left(\frac{1}{\|\bar{v}\|}\bar{v}\right)}_{(\text{Length})(\text{Direction})}$$



$\bar{u} - \text{proj}_{\bar{v}}\,\bar{u}$ is $\perp$ to $\bar{v}$!

This decomposes $\bar{u}$ into a component $\parallel$ (parallel) to $\bar{v}$ plus a component $\perp$ to $\bar{v}$.

If $\bar{u}$ & $\bar{v}$ aren't parallel

Then $\bar{u} - \text{proj}_{\bar{v}}\bar{u}$ & $\text{proj}_{\bar{v}}\bar{u}$ form a BASIS for $\mathbb{R}^2$ that is orthogonal.

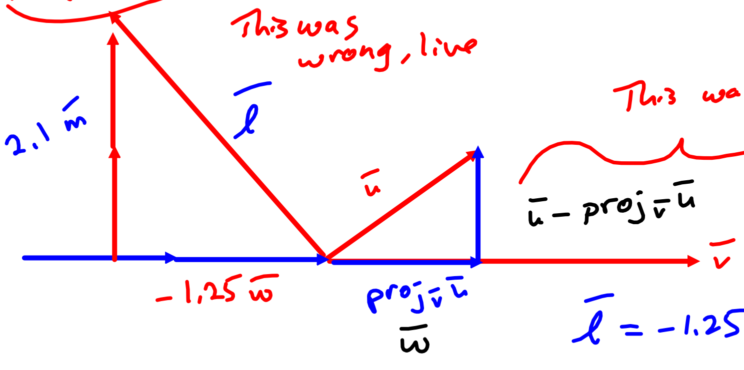
$\bar{i} = \langle 1, 0 \rangle$ & $\bar{j} = \langle 0, 1 \rangle$ comprise the CANONICAL BASIS for $\mathbb{R}^2$.

$\bar{u} = \langle u_1, u_2 \rangle = u_1 \langle 1, 0 \rangle + u_2 \langle 0, 1 \rangle$

Linear combo of $\bar{i}$ & $\bar{j}$.

You can do something similar with

$\frac{1}{\|\bar{u} - \text{proj}_{\bar{v}}\bar{u}\|}(\bar{u} - \text{proj}_{\bar{v}}\bar{u})$, $\frac{1}{\|\text{proj}_{\bar{v}}\bar{u}\|}\text{proj}_{\bar{v}}\bar{u}$ can do the same thing

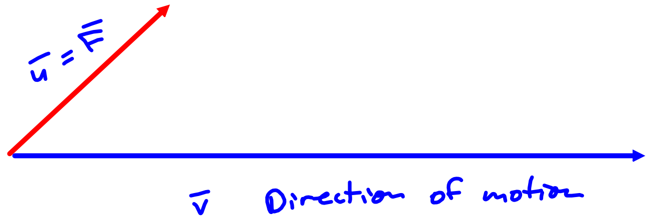


This was wrong, line

$\underbrace{\bar{u} - \text{proj}_{\bar{v}}\bar{u}}$ $= \bar{m}$

$\bar{\ell} = -1.25\,\bar{w} + 2.1\,\bar{m}$

Work:



Force Times Distance:

Need the component of the force vector that's parallel to the direction of the motion.

$$\text{Work} = \left(\|\text{proj}_{\bar{v}} \bar{F}\|\right)\left(\|\bar{v}\|\right)$$

$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\| \, \|\overrightarrow{PQ}\|$ or $W = \mathbf{F} \cdot \overrightarrow{PQ} = \bar{F} \cdot \bar{v}$

$\overrightarrow{PQ} = \bar{v}$

This is nice

$$\|\text{proj}_{\bar{v}} \bar{F}\| \, \|\bar{v}\|$$

$$= \left\|\left(\frac{\bar{F} \cdot \bar{v}}{\|\bar{v}\|}\right)\left(\frac{\bar{v}}{\|\bar{v}\|}\right)\right\| \, \|\bar{v}\|$$

$\|\text{This}\| = 1$

$$= \left\|\frac{\bar{F} \cdot \bar{v}}{\|\bar{v}\|}\right\| \, \|\bar{v}\| = \underbrace{\|\bar{F} \cdot \bar{v}\|}_{\text{No direction}}$$