Definition of Plane Curve

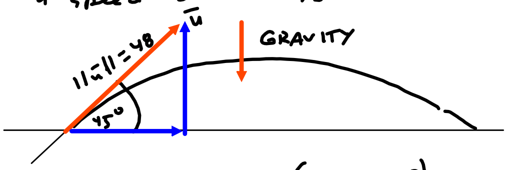
If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a **plane curve** C. The equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** for C, and t is the **parameter.**

Example

object propelled into the air at an angle of 45° at a speed of 48 ft/s.



$x = x(t) = (48\cos 45^\circ)t = 48 \cdot \frac{\sqrt{2}}{2} = \boxed{24\sqrt{2}\,t = x(t)}$

$y = y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$

$= -\frac{1}{2}(32)t^2 + (48\sin 45^\circ)t + 0$

$= -16t^2 + 48 \cdot \frac{\sqrt{2}}{2} t$

$\boxed{y(t) = -16t^2 + 24\sqrt{2}\,t}$

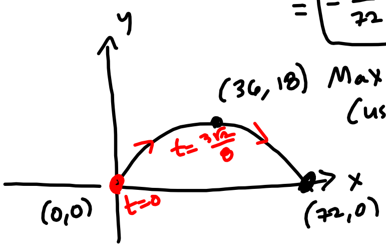
Eliminate the parameter: $x = 24\sqrt{2}\,t \Rightarrow$

$t = \frac{1}{24\sqrt{2}} x \Rightarrow$

$y = -16\left(\frac{1}{24\sqrt{2}} x\right)^2 + 24\sqrt{2}\left(\frac{1}{24\sqrt{2}} x\right)$

$= -16\left(\frac{1}{24^2 \sqrt{2}^2} x^2\right) + x$

$= \boxed{-\frac{1}{72}x^2 + x = y}$ A parabola!



(36, 18) Max
(using symmetry about $x = 36$
& intercepts)

$\frac{16}{24^2 \cdot 2} = \frac{\cancel{16}^{1}}{\cancel{24}_3 \cdot 24 \cdot \cancel{2}_1} = \frac{1}{3 \cdot 24} = \frac{1}{72}$

Find the peak:

$y = -\frac{1}{72}x^2 + x = -x\left(\frac{1}{72}x - 1\right) \overset{\text{SET}}{=} 0$

$x = 0$ $\qquad \frac{1}{72}x = 1$

$x = 72$

$\frac{72+0}{2} = 36 = x$

$\Rightarrow y = -\frac{1}{72}(36)^2 + 36 =$

THE PARAMETER tells us where we are, when.

$x = 0$
$t = \frac{1}{24\sqrt{2}}(0) = 0$

$\frac{\cancel{36}^{1} \cdot 36}{\cancel{72}_2} = \frac{1 \cdot 36}{2} = 18$

$x = 18$
$t = \frac{1}{24\sqrt{2}}(18) = \frac{3\sqrt{2}}{8}$

$\frac{\cancel{18}^{3}}{\cancel{24}_4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{4 \cdot 2} = \frac{3\sqrt{2}}{8}$

Sketching a Curve:

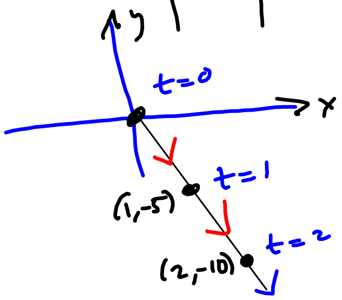
Sketching a Curve In Exercises 7–12, sketch and describe the orientation of the curve given by the parametric equations.

Direction of increasing t.

7. $x = t, \quad y = -5t$
8. $x = 2t - 1, \quad y = t + 4$
9. $x = t^2, \quad y = 3t$
10. $x = \sqrt{t}, \quad y = 2t - 1$
11. $x = 3\cos\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta \le \pi$
12. $x = \cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le 2\pi$

#7

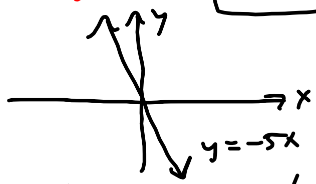
t	x	y
0	0	0
1	1	-5
2	2	-10



Eliminate parameter:

$x = t$

$y = -5t \Rightarrow \boxed{-5x = y}$



w/o the parameter we lose all sense of "when."

#11

$x = 3\cos\theta, \ y = 2\sin^2\theta$

Ouch! $\quad -3 \le x \le 3, \ 0 \le y \le 2$

$\frac{x}{3} = \cos\theta, \quad \frac{y}{2} = \sin^2\theta$

$\left(\frac{x}{3}\right)^2 = \cos^2\theta$

$\Rightarrow \left(\frac{x}{3}\right)^2 + \frac{y}{2} = \cos^2\theta + \sin^2\theta = 1$

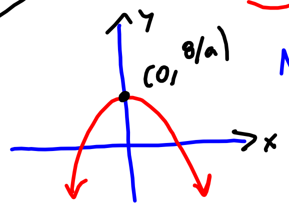
$\frac{x^2}{9} + \frac{y}{2} = 1$

$2x^2 + 9y = 18$ ← Aha! (Not "8")

$9y = -2x^2 + 18$

$y = -\frac{2}{9}x^2 + \frac{8}{9}$ → $\frac{18}{9} = 2$

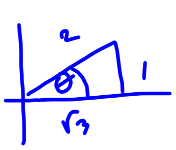
So, $y = -\frac{2}{9}x^2 + 2$



Not asked for, but helps my intuition.

Our graph will live inside this graph

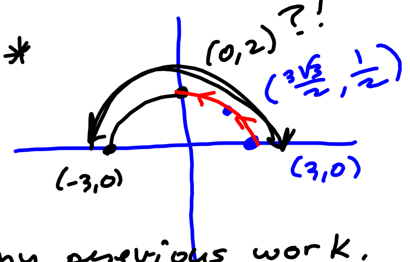
$\frac{\pi}{6} = 30^\circ$



$x = 3\cos\theta$

$y = 2\sin^2\theta$

θ	$x(\theta)$	$y(\theta)$
0	3	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0	2 *
$\frac{3\pi}{2}$	0	2



* what?! This contradicts my previous work.

Hmmm

Sketching a Curve In Exercises 13–38, (a) sketch the curve represented by the parametric equations (indicate the orientation of the curve) and (b) eliminate the parameter and write the resulting rectangular equation whose graph represents the curve. Adjust the domain of the rectangular equation, if necessary.

13. $x = t, \quad y = 4t$ **14.** $x = t, \quad y = -\frac{1}{2}t$

15. $x = -t + 1, \quad y = -3t$

16. $x = 3 - 2t, \quad y = 2 + 3t$

#13 $\boxed{y = 4x}$ (b)

#14 $\boxed{y = -\frac{1}{2}x}$ (b)

#16 $x = 3 - 2t = x$

$-2t = x - 3$

$t = \frac{x-3}{-2} = \frac{3}{2} - \frac{1}{2}x$

$\Rightarrow y = 2 + 3\left(\frac{3}{2} - \frac{1}{2}x\right)$

$= 2 + \frac{9}{2} - \frac{3}{2}x = \boxed{-\frac{3}{2}x + \frac{13}{2} = y}$ (b)

21. $x = \sqrt{t}, \quad y = 1 - t$ **22.** $x = \sqrt{t+2}, \quad y = t - 1$
23. $x = \sqrt{t} - 3, \quad y = t^3$
24. $x = \sqrt{t-1}, \quad y = \sqrt[3]{t-1}$
25. $x = t + 1, \quad y = \dfrac{t}{t+1}$ **26.** $x = t - 1, \quad y = \dfrac{t}{t-1}$
27. $x = 4\cos\theta, \quad y = 2\sin\theta$ **28.** $x = 2\cos\theta, \quad y = 3\sin\theta$
29. $x = 1 + \cos\theta, \quad y = 1 + 2\sin\theta$ **30.** $x = 2 + 5\cos\theta, \quad y = -6 + 4\sin\theta$
31. $x = 2\sec\theta, \quad y = \tan\theta, \quad \pi/2 \le \theta \le 3\pi/2$
32. $x = 3\cot\theta, \quad y = 4\csc\theta, \quad 0 \le \theta \le \pi$
33. $x = 3\cos\theta, \quad y = 3\sin\theta$ **34.** $x = 6\sin 2\theta, \quad y = 6\cos 2\theta$
35. $x = e^t, \quad y = e^{3t}$ **36.** $x = e^{-t}, \quad y = e^{3t}$
37. $x = t^3, \quad y = 3\ln t$ **38.** $x = \ln 2t, \quad y = 2t^2$

#21b $x = \sqrt{t}, \quad y = 1 - t$

$x = \sqrt{t} \Rightarrow t \ge 0$ & $x \ge 0$

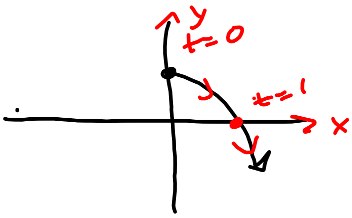
$\Rightarrow x^2 = t \Rightarrow$

$y = 1 - x^2$

Rect:

Parametric

t	x	y
0	0	1
1	1	0
4	2	-3



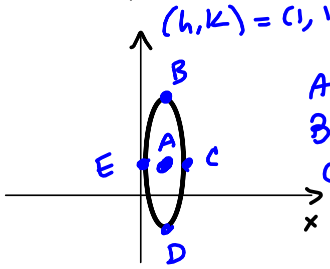
29 $x = 1 + \cos\theta \Rightarrow x - 1 = \cos\theta$

$y = 1 + 2\sin\theta \Rightarrow \frac{y-1}{2} = \sin\theta$

$\Rightarrow (x-1)^2 + \left(\frac{y-1}{2}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

$\frac{(x-1)^2}{1^2} + \frac{(y-1)^2}{2^2} = 1$ Ellipse!

$(h,k) = (1,1)$



$A = (1,1)$
$B = (1,3)$
$C = (2,1)$
$D = (1,-1)$
$E = (1,0)$

Doing Rectangular Graph.

Consider the following.

$x = 3 - 2t$
$y = 2 + 3t$

(a) Sketch the curve represented by the parametric equations (indicate the orientation of the curve).

(b) Eliminate the parameter and write the resulting rectangular equation whose graph represents the curve. Adjust the domain of the rectangular equation, if necessary.

I hate the table thing.

(b)

$x - 3 = -2t$

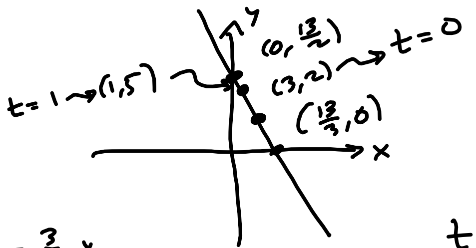
$t = \frac{x-3}{-2} = \frac{3}{2} - \frac{1}{2}x$

$y = 2 + 3\left(\frac{3}{2} - \frac{1}{2}x\right)$

$= 2 + \frac{9}{2} - \frac{3}{2}x = \frac{13}{2} - \frac{3}{2}x$

SET 0 $\Rightarrow \frac{13}{2} = \frac{3}{2}x$

$x = \frac{2}{3} \cdot \frac{13}{2} = \frac{13}{3}$



t	x	y
0	3	2
1	1	5

Graphing a Curve In Exercises 39–48, use a graphing utility to graph the curve represented by the parametric equations.

39. $x = t$, $y = \sqrt{t}$

40. $x = t + 1$, $y = \sqrt{2 - t}$

41. $x = 2t$, $y = |t + 1|$

42. $x = |t + 2|$, $y = 3 - t$

43. $x = 4 + 3\cos\theta$, $y = -2 + \sin\theta$

44. $x = 4 + 3\cos\theta$, $y = -2 + 2\sin\theta$

45. $x = 2\csc\theta$, $y = 4\cot\theta$

46. $x = \sec\theta$, $y = \tan\theta$

47. $x = \frac{1}{2}t$, $t = \ln(t^2 + 1)$

48. $x = 10 - 0.01e^t$, $y = 0.4t^2$

#46

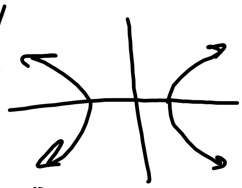
Recall:

$\tan^2\theta + 1 = \sec^2\theta$

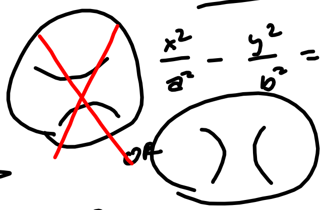
$x = \sec\theta$, $y = \tan\theta$ $\Longrightarrow$ $x^2 = y^2 + 1$

$\Longrightarrow x^2 - y^2 = 1$

Hyperbola!



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



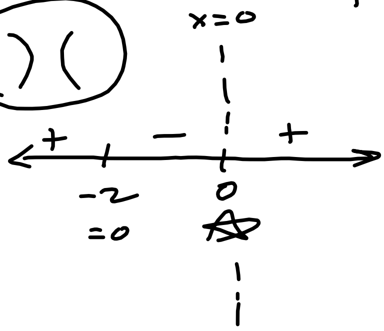
Consider the following.

$x = t - 2$

$y = \frac{t}{t - 2}$

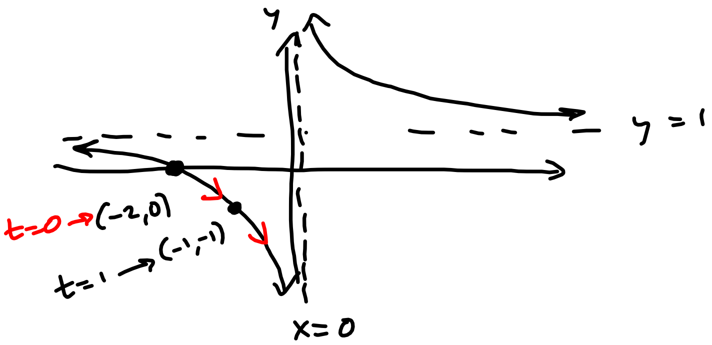
$\Longrightarrow t = x + 2 \Longrightarrow$

$y = \frac{x+2}{x+2-2} = \frac{x+2}{x}$



H.A.:

$\frac{x+2}{x} \xrightarrow{x \to \pm\infty} \frac{x}{x} = 1 = y$



t	x	y
0	−2	0
1	−1	$\frac{1}{-1} = -1$