§4.2

E4 $x^4-x^2-20=0$ Find all sol'ns.

Quadratic in form

u^2-u-20, where $u=x^2$

$(u-5)(u+4)=(x^2-5)(x^2+4)=0$

$x^2=5$ $x^2=-4$

$x=\pm\sqrt{5}$ $x=\pm\sqrt{-4}=\pm 2i$

FACTORED FORM:

$(x-\sqrt{5})(x+\sqrt{5})(x-2i)(x+2i)$

E5 Show $1+3i$ is a zero of $f(x)=x^4-3x^3+6x^2+2x-60$

(Show slicker way of multiplying $(x-(a+bi))(x-(a-bi))$

Avoids Synthetic Division by $x-(1+3i)$

(I'd USE " ")

Book does this: Recall $(a+bi)(a-bi)=a^2+b^2$

$$(x-(1+3i))(x-(1-3i)) = ((x-1)-3i)((x-1)+3i) = (x-1)^2+3^2$$

$$(a-bi)(a+bi)$$

$= x^2-2x+1+9 = x^2-2x+10$. Then long division to factor out $x^2-2x+10$:

$$\begin{array}{r l}
 & x^2 - x - 6 \\
x^2-2x+10 & \overline{)\,x^4-3x^3+6x^2+2x-60} \\
 & -(x^4-2x^3+10x^2) \\
 & \quad -x^3-4x^2+2x-60 \\
 & -(-x^3+2x^2-10x) \\
 & \quad -6x^2+12x-60 \\
 & -(-6x^2+12x-60) \\
 & \quad 0
\end{array}$$

$\frac{x^4}{x^2}$

$\frac{-x^3}{x^2}=-x$

$\frac{-6x^2}{x^2}$

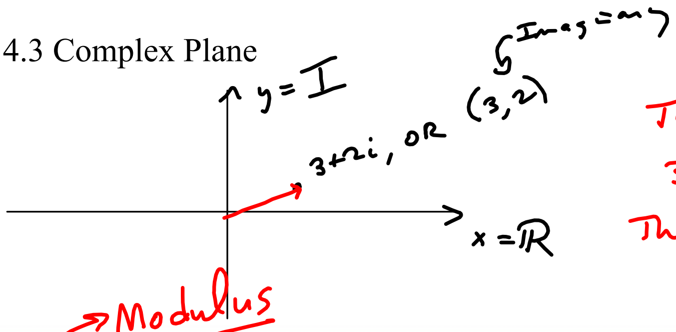
$x^4-3x^3+6x^2+2x-60$

$= (x-(1+3i))(x-(1-3i))(x^2-x-6)$

$= \boxed{(x-(1+3i))(x-(1-3i))(x-3)(x+2)}$

→ Split into linear factors.

S 4.3 Complex Plane



Think of 3+2i as <3,2>.

Then $|3+2i| = \sqrt{3^2+2^2}$

Modulus

Definition of the Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

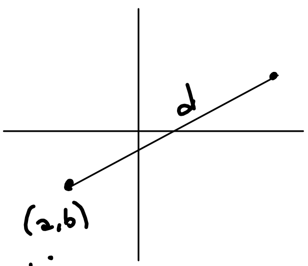
$|a + bi| = \sqrt{a^2 + b^2}$. Pythagorus! Distance in the real plane!

Length/magnitude of a vector!

Distance Formula in the Complex Plane

The distance d between the points (a, b) and (s, t) in the complex plane is

$$d = \sqrt{(s - a)^2 + (t - b)^2}.$$



$(s,t) = s+ti = (x_2, y_2)$

$= a+bi$

$= (x_1, y_1)$

$$d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

$$= \sqrt{(a-s)^2 + (b-t)^2}$$

OR $\sqrt{(s-a)^2 + (t-b)}$

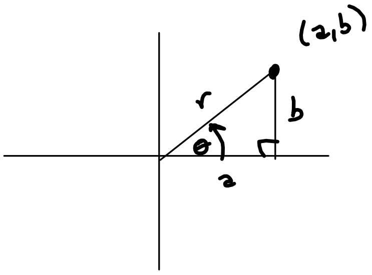
S 4.4 Trigonometric Form of a Complex Number

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2+b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.



$\frac{b}{r} = \sin\theta$

$b = y = r\sin\theta$

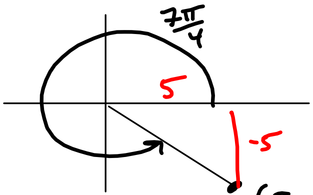
$a = x = r\cos\theta$

$r = \text{modulus} = \sqrt{a^2+b^2}$

$z = 5 - 5i$

$|z| = \sqrt{5^2+5^2}$

$= \sqrt{2\cdot 25} = 25\sqrt{2}$ → 5



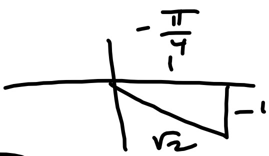
(5,−5) or 5−5i

$\tan\theta = -1$

$\boxed{\theta = -\frac{\pi}{4}}$ or $\frac{7\pi}{4}$ or . . .

→ arctan(−1), but it's not the only sol'n of

$\tan\theta = \frac{y}{x} = \frac{-5}{5} = -1$



This says

$z = 25\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ (25 → 5)

or $25\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$ (25 → 5)

$= 25\sqrt{2}\left(\left(\frac{1}{\sqrt{2}}\right) + i\left(-\frac{1}{\sqrt{2}}\right)\right)$ (25 → 5)

$= (5\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + 5\sqrt{2}\left(-\frac{1}{\sqrt{2}}i\right) = \boxed{5 - 5i}$

Multiplication of Complex Numbers in Trigonometric Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Recall $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\left(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + \underbrace{i^2\sin\theta_1\sin\theta_2}_{-\sin\theta_1\sin\theta_2}\right)$$

$$= r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right]\right)$$

$$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

Holy smokes! The angles add!

The moduli multiply!

To Rotate a vector in the plane by an angle θ, just multiply it as a complex # by $\underline{\cos\theta + i\sin\theta}$

Vectors $\longleftrightarrow$ Complex #s

Structurally identical, except Complex #s do some funky multiplication stuff.

vectors don't really multiply & give you another vector.

§4.5 (4.5 questions on 4.4 homework!?!)
(Change of editions to 10^{th} edition)

DeMoivre's Theorem
Special Case of Multiplying:

POWERS

$z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$

$z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$

etc. $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

ROOTS

$x^4 - 1 = 0$

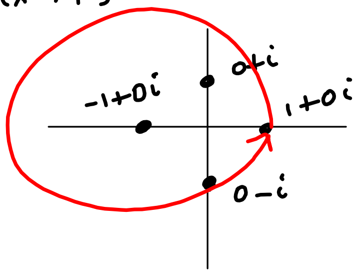
$u^2 - 1 = 0 \quad (u = x^2)$

$(u-1)(u+1)$

$= (x^2-1)(x^2+1) = (x-1)(x+1)(x-i)(x+i)$

$x^2 = -1$

$x = \pm\sqrt{-1} = \pm i$



4^{th} roots of 1.

$1^4 = 1$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$(-1)^4 = 1$

$(-i)^4 = (-1)^4(i)^4 = 1$

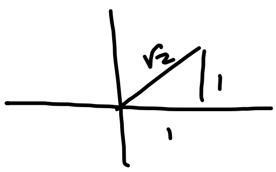
They're all 4^{th} roots of 1!

They're evenly distributed around the circle, i.e., separated by $\frac{2\pi}{4} = \frac{\pi}{2}$ radians.

Find all 4^{th} roots of $1+i$

4^{th} roots: increment $= \frac{2\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



Principal 4^{th} root is $\sqrt[4]{1+i}$ & has an argument that's $\frac{1}{4}$ the argument of $1+i$.

$$\frac{\frac{\pi}{4}}{4} = \frac{\pi}{4}\cdot\frac{1}{4} = \frac{\pi}{16}$$

The next one is $\frac{2\pi}{4} = \frac{\pi}{2}$ radians counterclockwise from it:

$$\frac{\pi}{16} + \frac{2\pi}{4}\cdot\frac{4}{4} = \frac{\pi}{16} + \frac{8\pi}{16} = \frac{9\pi}{16}$$

$$\frac{9\pi}{16} + \frac{8\pi}{16} = \frac{17\pi}{16}$$

$$\frac{(17+8)\pi}{16} = \frac{25\pi}{16}$$

$\frac{(25+8)\pi}{16} = \frac{33\pi}{16} = \frac{32\pi}{16} + \frac{\pi}{16}$ is coterminal with $\frac{\pi}{16}$. (Went too far)

Summarize the work:

$\frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, \frac{25\pi}{16}$. Need the length!

$\sqrt[4]{\text{length}}$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$, so we need the 4^{th} root of that.

$$\sqrt[4]{\sqrt{2}} = \left(2^{\frac{1}{2}}\right)^{\frac{1}{4}} = 2^{\frac{1}{2}\cdot\frac{1}{4}} = 2^{\frac{1}{8}} = \sqrt[8]{2}$$

$\Rightarrow$ 4^{th} roots of $1+i$ are

$$\sqrt[8]{2}\left(\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right)$$

$$\sqrt[8]{2}\left(\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right)$$

$$\sqrt[8]{2}\left(\cos\left(\frac{17\pi}{16}\right) + i\sin\left(\frac{17\pi}{16}\right)\right)$$

$$\sqrt[8]{2}\left(\cos\frac{25\pi}{16} + i\sin\left(\frac{25\pi}{16}\right)\right)$$

n^{th} roots of $z = r(\cos\theta + i\sin\theta)$ are

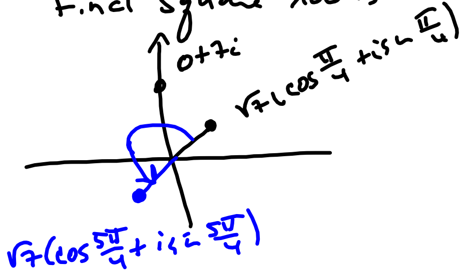
$$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right), \quad k = 0, 1, 2, \dots, n-1$$

$r^{\frac{1}{n}} \leftrightarrow \sqrt[n]{r}$

I'd just remember $\frac{\theta}{n}$, $\frac{2\pi}{n}$ & add $\frac{2\pi}{n}$ to $\frac{\theta}{n}$ & keep adding it, 'til done. (i.e., $n-1$ times)

WebAssign #9

Find square roots of $7i$



$$\frac{\pi}{4} + \frac{2\pi}{2}\cdot\frac{2}{2} = \frac{\pi + 4\pi}{4} = \frac{5\pi}{4}$$

$7i$, $\theta = \frac{\pi}{2}$, $|7i| = 7$

$n = 2$, so

$\frac{\theta}{2} = \frac{\pi}{4}$, $\sqrt{7} = 7^{\frac{1}{2}} = \sqrt[2]{7}$

$\sqrt{7}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) =$

$\sqrt{7}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

WebAssign
#11

$z = 216\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$

Find cube roots

$\frac{\theta}{3} = \frac{5\pi}{6} \cdot \frac{1}{3} = \frac{5\pi}{18}$

Increment: $\frac{2\pi}{3} = \frac{12\pi}{18}$

$6\left(\cos\left(\frac{5\pi}{18}\right) + i\sin\left(\frac{5\pi}{18}\right)\right)$

$6\left(\cos\left(\frac{17\pi}{18}\right) + i\sin\left(\frac{17\pi}{18}\right)\right)$

$6\left(\cos\left(\frac{29\pi}{18}\right) + i\sin\left(\frac{29\pi}{18}\right)\right)$

$$
\begin{array}{r|l}
2 & 216 \\
2 & 108 \\
2 & 54 \\
3 & 27 \\
3 & 9 \\
 & 3
\end{array}
$$

$\sqrt[3]{216} = 2 \cdot 3 = 6$