Remind me to hit 'record.'

Today: Prove Law of Cosines

Use Law of cosines to prove that $\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}$, where θ = angle between 2 vectors $\bar{u}, \bar{v}$.

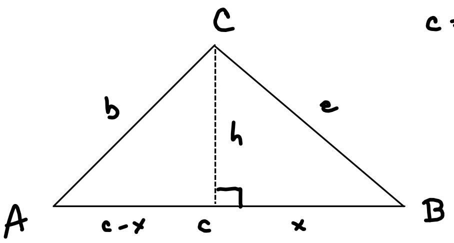
Examples

wP#2 4/1

10	Sec 3.4 WP#3, Due 3/30 Test 3 over Chapters 1 - 3, Open 3/31 - 4/3 Sec 4.1	3/28

From schedule (which is kinda messed-up):

https://harryzaims.com/122/122-spring-22/122-schedule-spring-22.pdf



$c = x + (c-x)$

NOTE

$\frac{h}{a} = \sin B$ True. No help!

$h = a\sin B$

$\frac{h}{b} = \sin A$

$h = b\sin A$ Yes!

$\frac{c-x}{b} = \cos A$

$c - x = b\cos A$

$-x = -c + b\cos A$

$x = c - b\cos A$

Want to prove:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Pf: $a^2 = x^2 + h^2$

$= (c - b\cos A)^2 + (a\sin B)^2$ Not using $(b\sin A)^2$

$= c^2 - 2bc\cos A + (b\cos A)^2 + b^2\sin^2 A$

RECALL

$(r-s)^2 = r^2 - 2rs + s^2$

$(r-s)(r-s) = r^2 - rs - rs + s^2$
$= r^2 - 2rs + s^2$

$= c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A$

$= c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A)$

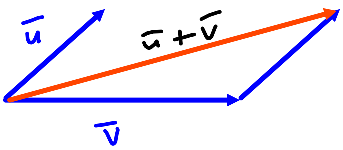
$= c^2 - 2bc\cos A + b^2 = \boxed{b^2 + c^2 - 2bc\cos A = a^2}$

Questions from §3.1, 3.2?

Book gives a form where $\cos A$ is solved-for.

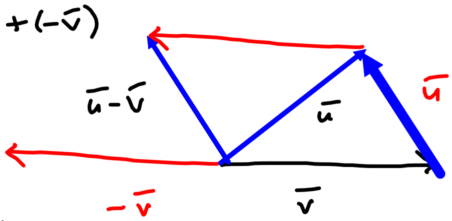
I just remember the basic formula & manipulate it on-the-fly.

Let's talk about vectors



$\bar{u}+\bar{v}$ is the diagonal of the parallelogram defined by $\bar{u}$ & $\bar{v}$.

$\bar{u}-\bar{v} = \bar{u}+(-\bar{v})$



$\bar{u}-\bar{v}$ also: Points from the tip of $\bar{v}$ to the tip of $\bar{u}$.

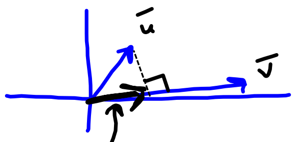
Now, Dot Product & Length/Magnitude of and angles between vectors. $\cos\theta$

$\bar{u} = \langle u_1, u_2 \rangle$, $\bar{v} = \langle v_1, v_2 \rangle \implies$

$\bar{u}\cdot\bar{v} = u_1v_1 + u_2v_2$

It's a measure of how "parallel" $\bar{u}$ & $\bar{v}$ are.

$\bar{u} = \langle 3,4 \rangle$, $\bar{v} = \langle 1,5 \rangle$



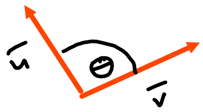
$\bar{u}\cdot\bar{v} = 3\cdot 1 + 4\cdot 5 = 3+20 = 23$

In a sense, this is measuring the length of the shadow cast by $\bar{u}$ on $\bar{v}$

The shadow of $\bar{u}$ on $\bar{v}$.

Light source perpendicular to $\bar{v}$.

$\perp$ means perpendicular or "orthogonal."



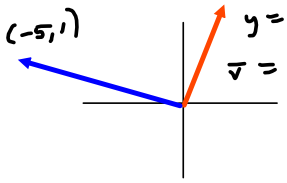
$\bar{u}$ & $\bar{v}$ $\perp$ to each other

No shadow!

0 shadow

$\theta = 90^\circ \Rightarrow 0$ shadow

Note: $\cos(90^\circ) = 0$!



Scratch:
$y = 5x$
$\bar{v} = \langle 1,5 \rangle$

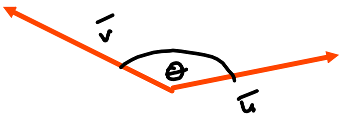
$m = 5 \Rightarrow m_\perp = -\frac{1}{5}$
so $y = -\frac{1}{5}x$
$\bar{u} = \langle -5,1 \rangle$

I MADE $\bar{u} \perp \bar{v}$.

NOTE $\bar{u}\cdot\bar{v} = \langle 1,5 \rangle \cdot \langle -5,1 \rangle$
$= (1)(-5) + (5)(1) = 0$!

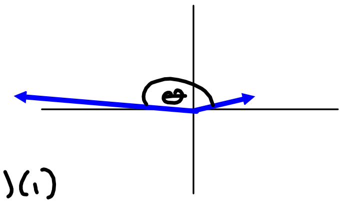
$\bar{u}\cdot\bar{v} = 0$ when $\bar{u} \perp \bar{v}$!

$\theta > 90°$



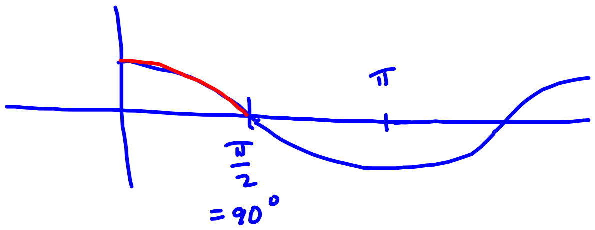
$\bar{v} = \langle 5, 1 \rangle$

$\bar{u} = \langle -10, 1 \rangle$



$\bar{u} \cdot \bar{v} = (-10)(5) + (1)(1)$

$= -50 + 1 = -49 < 0$ when $\theta > 90°$!



Compare the graph of cosine!

$0 \le \theta < 90 \Rightarrow \cos\theta > 0$

$90 < \theta \le 180 \Rightarrow \cos\theta < 0$

Just Like the dot product!

In fact,

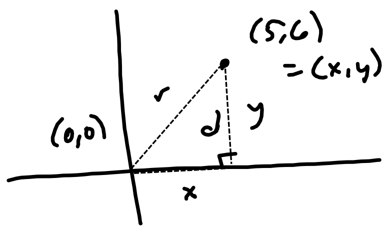
$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$

I will prove shortly.

But first,

$$\|\bar{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{\bar{u} \cdot \bar{u}} \;!$$

Find the distance from (5,6) to the origin:



Distance Between (x_1, y_1) & (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$x^2 + y^2 = r^2$

$(x_1, y_1) = (0, 0)$

$(x_2, y_2) = (5, 6)$

$$\Rightarrow d = \sqrt{(0-5)^2 + (0-6)^2}$$

$$= \sqrt{5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}$$

The length of
$\bar{u} = \langle 5,6 \rangle$ is the distance from $(0,0)$ to the tip of $\bar{u}$,
i.e., the distance from $(0,0)$ to $\langle 5,6 \rangle$

$$\|\bar{u}\| = \sqrt{5^2+6^2} = \sqrt{5\cdot 5 + 6\cdot 6} = \sqrt{\bar{u}\cdot\bar{u}}$$

So there's the connection between dot product & length.

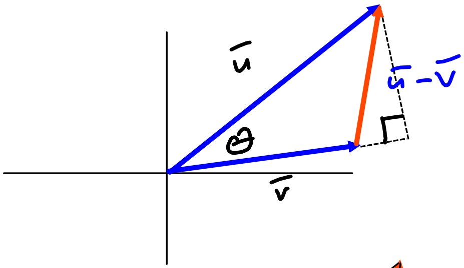
Properties of dot product:

$\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$ (commutative)

$\langle 1,2 \rangle = \bar{u}$, $\bar{v} = \langle -3,-10 \rangle$

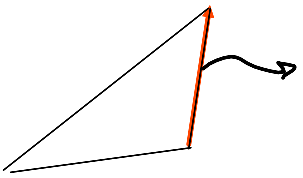
$$\Rightarrow \bar{u}\cdot\bar{v} = (1)(-3)+(2)(-10) = (-3)(1) + (-10)(2)$$
$$= \bar{v}\cdot\bar{u}.$$

Let's talk about the angle θ between $\bar{u}$ & $\bar{v}$

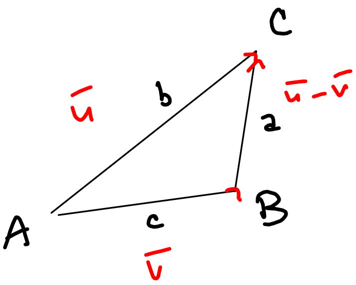


We use Law of Cosines to prove

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$



? = $\bar{u}-\bar{v}$!



Law of cosines says:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$a = \|\bar{u}-\bar{v}\|$
$b = \|\bar{u}\|$
$c = \|\bar{v}\|$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\|\bar{u}-\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos A$$

$$(\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v}) = \bar{u}\cdot\bar{u} - \bar{u}\cdot\bar{v} - \underline{\bar{v}\cdot\bar{u}} + (-\bar{v})\cdot(-\bar{v})$$

$$= \bar{u}\cdot\bar{u} - 2\bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v}$$

$$= \underline{\|\bar{u}\|^2} - 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2 = \underline{\|\bar{u}\|^2} + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos A$$

$$\Rightarrow\quad -2\bar{u}\cdot\bar{v} = -2\|\bar{u}\|\|\bar{v}\|\cos A = -2\bar{u}\cdot\bar{v}$$

$$\Rightarrow\quad \boxed{\cos A = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}}$$

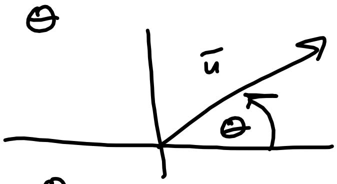
→ Most commonly used.

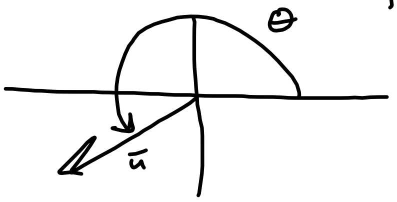
I just remember this ⤴, but here's a common version of it:

$$\boxed{\bar{u}\cdot\bar{v} = \|\bar{u}\|\|\bar{v}\|\cos\theta}$$

Can be handy.

Direction Angle θ





Angle formed by
$\bar{u}$ & positive x-axis

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unit vectors: vectors of length "1."

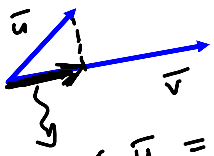
Build them easily:

$\bar{u} = \langle u_1, u_2 \rangle \longrightarrow$

$\frac{1}{\|\bar{u}\|}\bar{u}$ is unit vector in direction of $\bar{u}$.

Next time: Finish C3 theory:

$\bar{w} = proj_{\bar{v}}\bar{u}$ = projection of $\bar{u}$ on $\bar{v}$ is the "shadow vector

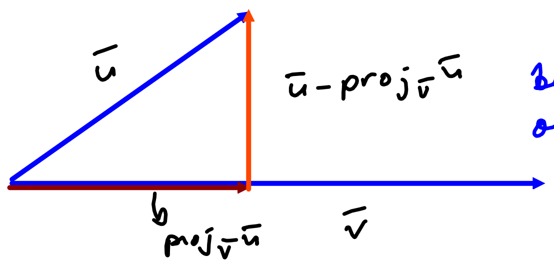


$proj_{\bar{v}}\bar{u}$ = shadow cast on $\bar{v}$ by $\bar{u}$

$\|proj_{\bar{v}}\bar{u}\| = comp_{\bar{v}}\bar{u}$ = component of $\bar{u}$ in the direction of $\bar{v}$

We'll decompose $\bar{u}$ into $proj_{\bar{v}}\bar{u} + (\bar{u} - proj_{\bar{v}}\bar{u})$

Why? Because that writes $\bar{u}$ as the sum of 2 vectors: one $\perp$ to $\bar{v}$ & one $\parallel$ to $\bar{v}$



This is the idea behind Gram-Schmidt orthogonalization.

CALC III & Linear Algebra.