2. Let $z=-\sqrt{3}-i$

a. (10 pts) Find $z+\bar{z}$ and $z\bar{z}$, where $\bar{z}$ is the complex conjugate of z.

b. (10 pts) Express z in trigonometric form.

$a+bi$

(a) $z+\bar{z} = -\sqrt{3} - i + -\sqrt{3} + i$

$= -2\sqrt{3} = 2a$ (2a)

$z\bar{z} = a^2+b^2 =$

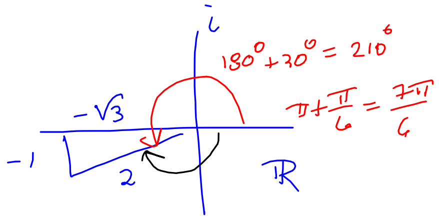
$(-\sqrt{3}-i)(-\sqrt{3}+i) = (-\sqrt{3})(-\sqrt{3}) \underline{-\sqrt{3}\,i + \sqrt{3}\,i} - i^2$ (2a)

$(a+b)(a-b) = a^2-b^2$ $\qquad = 3-(-1) = 3+1 = \boxed{4 = z\bar{z}}$

$(a+bi)(a-bi) = a^2-(bi)^2 = a^2-b^2i^2 = a^2+b^2$

$\sqrt{3}\sqrt{3} = \sqrt{3\cdot 3} = 3$

Recall $\|z\| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}} = \sqrt{4} = \boxed{2 = \|z\|}$



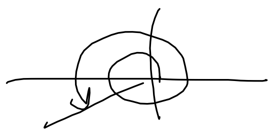
(2b) Trig form $\|z\|(\cos\theta + i\sin\theta)$

The representation isn't unique, but we're looking for this guy

$= 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 2\left(\cos\left(\frac{19\pi}{6}\right) + i\sin\left(\frac{19\pi}{6}\right)\right)$



3. Let $z = 8\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

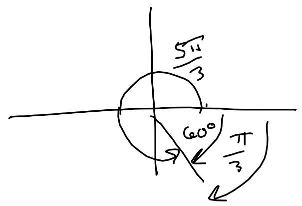
a. (10 pts) Express z in standard form.

b. (10 pts) Find the principal 3rd root of z, i.e., find $\sqrt[3]{z}$. Leave z in trigonometric form for this.

c. (10 pts) Now, find the *other* 3rd roots of z, in trigonometric form.

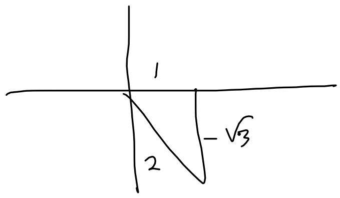
d. (10 pts) Find the trigonometric form of z^2.

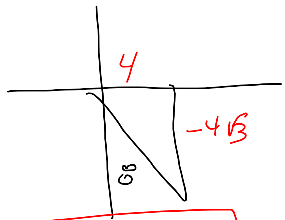
a



$\frac{5\pi}{3} = \frac{3\pi}{3} + \frac{2\pi}{3} = \pi + \frac{2\pi}{3}$

$\frac{5\pi}{3} \cdot \frac{180°}{\pi} = 5 \cdot 60° = 300°$





$z = 8\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= 8\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= 4 - 4\sqrt{3}\,i$

$\boxed{z = 4 - 4\sqrt{3}\,i}$

---

b

$\sqrt[3]{z} = \sqrt[3]{\|z\|}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$

$= \boxed{2\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right) = \sqrt[3]{z}}$

$\frac{\frac{5\pi}{3}}{3} = \frac{\frac{5\pi}{3}}{\frac{3}{1}} = \frac{5\pi}{3} \cdot \frac{1}{3}$

$= \frac{5\pi}{9}$

c The other 3rd roots:

Increment $= \frac{2\pi}{3}$

$\frac{5\pi}{9} + \frac{2\pi}{3} \cdot \frac{3}{3} = \frac{5\pi}{9} + \frac{6\pi}{9} = \frac{11\pi}{9}$

$+ \frac{6\pi}{9} = \frac{17\pi}{9} + \frac{6\pi}{9} = \frac{23\pi}{9}$

No!

$\frac{11\pi}{9} + \frac{6\pi}{9} = \frac{17\pi}{9}$

$\frac{17\pi}{9} + \frac{6\pi}{9} = \frac{23\pi}{9}$

$2\left(\cos\left(\frac{11\pi}{9}\right)+i\sin\left(\frac{11\pi}{9}\right)\right),$

$2\left(\cos\left(\frac{17\pi}{9}\right)+i\sin\left(\frac{17\pi}{9}\right)\right),$

$\boxed{2\left(\cos\left(\frac{23\pi}{9}\right)+i\sin\left(\frac{23\pi}{9}\right)\right)}$

Redundant! Fool!

$\frac{23\pi}{9}=\frac{18\pi}{9}+\frac{5\pi}{9}=2\pi+\frac{5\pi}{9}$

is coterminal with $\frac{5\pi}{9}$

(d) $z^2 = 8^2\left(\cos\left(2\cdot\frac{5\pi}{3}\right)+i\sin\left(2\cdot\frac{5\pi}{3}\right)\right)$

$= 64\left(\cos\left(\frac{10\pi}{3}\right)+i\sin\left(\frac{10\pi}{3}\right)\right)$

$\sqrt[n]{z}=z^{\frac{1}{n}}$

$z^n = \|z\|^n\left(\cos(n\theta)+i\sin(n\theta)\right)$

$\sqrt[n]{z}=\sqrt[n]{z}\left(\cos\left(\frac{1}{n}\theta\right)+i\sin\left(\frac{1}{n}\theta\right)\right)$

4. (10 pts) Solve $3\csc^3(2\theta) - 6\csc^2(2\theta) - \csc(2\theta) + 2 = 0$.

(Hint: If $f(x) = 3x^3 - 6x^2 - x + 2$, then $f(2) = 0$.)

$2\,|\,12$, $2\,|\,6$, 3

Brilliant Job on wrong prob.

$\boxed{\text{Missed the "}2\theta\text{"}}$

$$\begin{array}{r|rrrr} 2 & 3 & -6 & -1 & 2 \\ & & 6 & 0 & -2 \\ \hline & 3 & 0 & -1 & 0 \\ & x^2 & x & c & r \end{array}$$

$(x-2)(3x^2-1) \overset{\text{SET}}{=} 0$

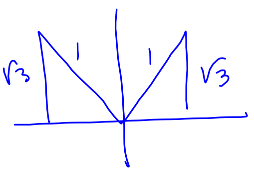
$(\sqrt{3}x-1)(\sqrt{3}x+1)(x-2)$

$\sqrt{3}x - 1 = 0$ $\quad \sqrt{3}x + 1 = 0$ $\quad x - 2 = 0$

$\sqrt{3}x = 1$ $\quad \sqrt{3}x = -1$ $\quad x = 2$

$x = \frac{1}{\sqrt{3}}$ $\quad x = -\frac{1}{\sqrt{3}}$

$\csc x = \frac{1}{\sqrt{3}} \Rightarrow \sin x = \sqrt{3}$



ugh! Not a nice, pretty triangle!

UNPOSSIBLE!

No solutions from the $3x^2-1$ factor!

$3x^2 - 1 = 0$

$a = 3, b = 0, c = -1$

$b^2 - 4ac = 0^2 - 4(3)(-1) = 12$

$\sqrt{12} = 2\sqrt{3}$

$x = \frac{-0 \pm 2\sqrt{3}}{2(3)} = \pm\frac{2\sqrt{3}}{6} = \pm\frac{\sqrt{3}}{3}$

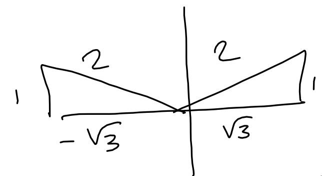
$(x-2)(3x^2-1) \overset{\text{SET}}{=} 0$

$3x^2 - 1 = 0$

$3x^2 = 1$

$x^2 = \frac{1}{3} \Rightarrow x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$

So $\csc\theta = 2 \Rightarrow \sin\theta = \frac{1}{2}$



$$\boxed{\text{So, } x \in \left\{\frac{5\pi}{6}, \frac{\pi}{6}\right\} = \{150^\circ, 30^\circ\}}$$

4. (10 pts) Solve $3\csc^3(2\theta) - 6\csc^2(2\theta) - \csc(2\theta) + 2 = 0$.

(Hint: If $f(x) = 3x^3 - 6x^2 - x + 2$, then $f(2) = 0$.)

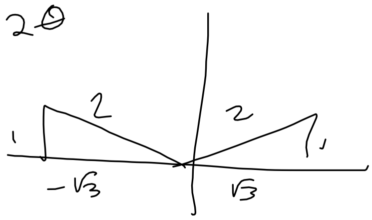
On YOUR test, I will specify looking for solutions in $[0, 2\pi)$.

Then the followup will be "Find ALL solutions," when you basically add $2n\pi$, $n \in \mathbb{Z}$ to the solutions.

Handling the "2θ"

$0 \le \theta < 2\pi$

$0 \le 2\theta < 4\pi$



So

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

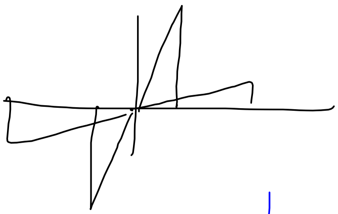
$$\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$$

This means

$$\theta \in \left\{ \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12} \right\}$$

covers all of $[0, 2\pi)$

The more general "Find ALL solutions":



$$\theta \in \left\{ x + 2n\pi \;\middle|\; x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \forall n \in \mathbb{Z} \right\}$$

$$= \left\{ x + n\pi \;\middle|\; x = \frac{\pi}{12}, \frac{5\pi}{12}, \forall n \in \mathbb{Z} \right\}$$

$$= \left\{ x + 360^\circ n \;\middle|\; x = 15^\circ, 75^\circ, 195^\circ, 255^\circ, \forall n \in \mathbb{Z} \right\}$$

$$= \left\{ x + 180^\circ n \;\middle|\; x = 15^\circ, 75^\circ, \forall n \in \mathbb{Z} \right\}$$

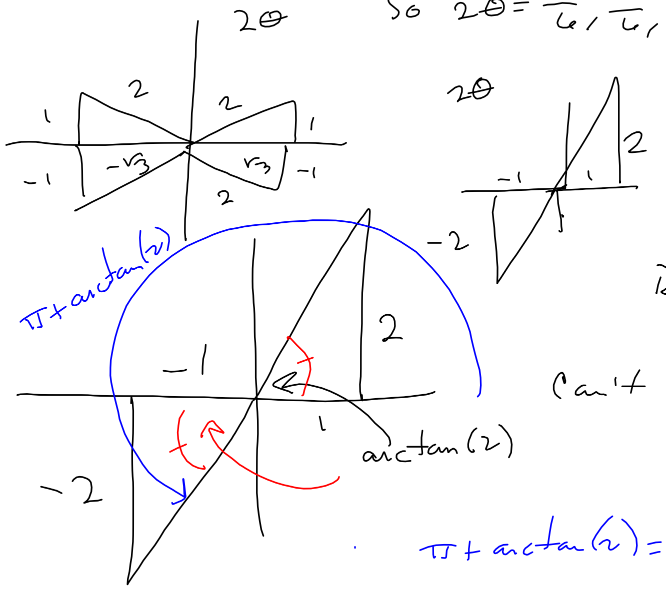
4. (10 pts) Solve $3\csc^3(2\theta) - 6\csc^2(2\theta) - \csc(2\theta) + 2 = 0$.

(Hint: If $f(x) = 3x^3 - 6x^2 - x + 2$, then $f(2) = 0$.)

Change the "csc" to "tan"

$\tan(2\theta) = \pm\frac{1}{\sqrt{3}}$, $\qquad 0 \le \theta \le 4\pi$

So $2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{4}, \frac{19\pi}{6}, \frac{23\pi}{6},$



$\tan 2\theta = 2$

Not a nice clean answers.

Rather: arctan(2), arctan(2)+π

Can't make it any cleaner

π + arctan(2) = π + tan⁻¹(2)
↑ calculator.

arctan(2) + 2π,
arctan(2) + π + 2π

180° + arctan(2)
↑ in degrees.

So 2θ = arctan(2), arctan(2)+π, arctan(2)+2π,
arctan(2) + 3π

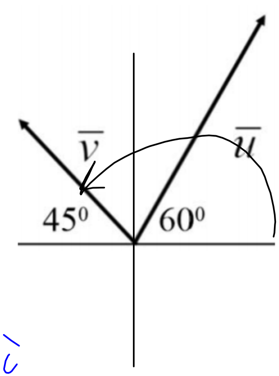
Ugh! Changing to tan from csc opened up more answers & more work!

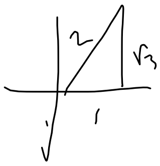
∴ $\theta = \frac{\arctan(2)}{2}, \frac{\arctan(2)+\pi}{2}, \frac{\arctan(2)+2\pi}{2}, \frac{\arctan(2)+3\pi}{2}$

Bonus 3. Dad's out walking his dog and his toddler. The dog pulls with 50 pounds of force in the direction of the vector $\overline{u}$. The toddler pulls with 30 pounds of force in the direction of the vector $\overline{v}$.

a. (5 pts) Express $\overline{u}$ and $\overline{v}$ in component form, in two ways: Give an exact answer, and an answer rounded to 3 decimal places.

b. (5 pts) What's the net force, as a vector, on poor Dad? Give an exact answer, and an answer rounded to 3 decimal places.



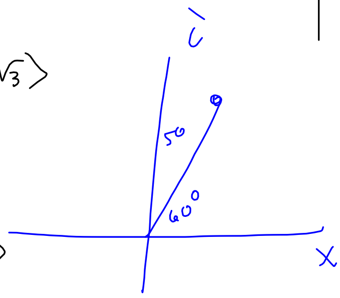


$\overline{u} = 50\langle \cos 60^\circ, \sin 60^\circ \rangle$

(a) $= 50\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle 25, 25\sqrt{3} \rangle$

$\overline{v} = 30\langle \cos 135^\circ, \sin(135^\circ) \rangle$

$= 30\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle \frac{-30}{\sqrt{2}}, \frac{30}{\sqrt{2}} \rangle$



$50(\cos 60^\circ + i \sin 60^\circ)$

(b) $\overline{u} + \overline{v} = \langle 25 - \frac{30}{\sqrt{2}}, 25\sqrt{3} + \frac{30}{\sqrt{2}} \rangle$