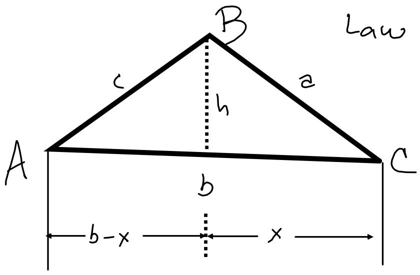


Law of Cosines. We prove $c^2 = a^2+b^2-2ab\cos C$:

$c^2 = h^2 + (b-x)^2$,

$\frac{x}{a} = \cos C \Rightarrow x = a\cos C$,

$\frac{h}{a} = \sin C \Rightarrow h = a\sin C$, $\rightarrow$

$$\Rightarrow c^2 = (a\sin C)^2 + (b - a\cos C)^2$$
$$= a^2\sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C$$
$$= a^2\sin^2 C + a^2\cos^2 C + b^2 - 2ab\cos C$$
$$= a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C$$
$$= a^2+b^2-2ab\cos C \ \blacksquare$$

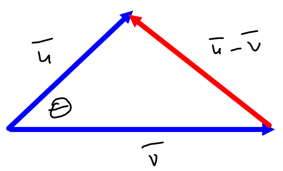
Dot Product: $\langle x,y\rangle \circ \langle z,w\rangle = xz+yw$ $\quad \langle a,b\rangle \cdot \langle a,b\rangle = a^2+b^2$

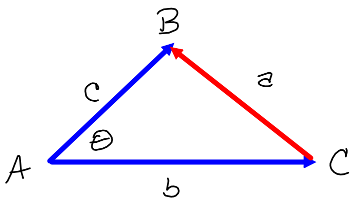
$\langle 2,3\rangle \circ \langle -5,6\rangle = -10+18 = 8$

Magnitude $\|\bar{u}\| = \sqrt{a^2+b^2} = \sqrt{\bar{u}\circ\bar{u}}$!

$\bar{u} = \langle a,b\rangle \Rightarrow$ ↗ Dot product's connection to length !

Dot product's connection to ANGLES.





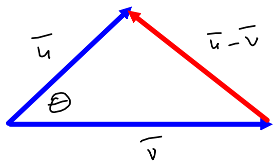
$A = \Theta$

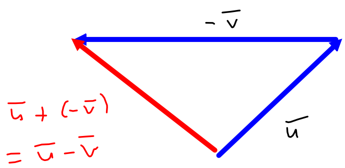
want θ, via cosine, and cosine via dot product.

$$a^2 = b^2+c^2-2bc\cos A$$
$$\|\bar{u}-\bar{v}\|^2 = \|\bar{v}\|^2 + \|\bar{u}\|^2 - 2\|\bar{v}\|\|\bar{u}\|\cos\Theta$$
$$(\bar{u}-\bar{v})\circ(\bar{u}-\bar{v}) = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos\Theta$$
$$\bar{u}\cdot\bar{u} - \bar{u}\cdot\bar{v} - \bar{v}\circ\bar{u} + \bar{v}\circ\bar{v} = \nearrow$$
$$= \|\bar{u}\|^2 - 2\bar{u}\circ\bar{v} + \|\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos\Theta$$
$$\Rightarrow -2\bar{u}\circ\bar{v} = -2\|\bar{u}\|\|\bar{v}\|\cos\Theta \Rightarrow$$
$$\bar{u}\cdot\bar{v} = \|\bar{u}\|\|\bar{v}\|\cos\Theta \Rightarrow$$
$$\boxed{\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \cos\Theta} \ \blacksquare$$

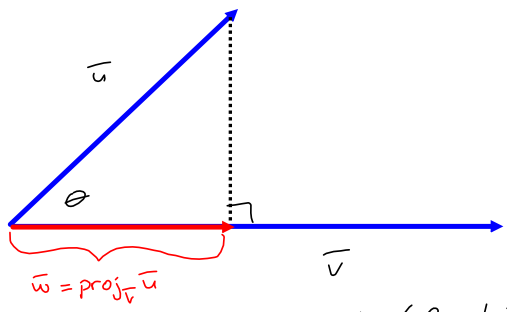
↳ was on the cheat sheet

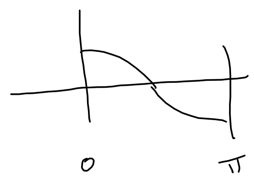
$$\text{comp}_{\bar{v}}\bar{u} = \frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|}$$
$$\text{proj}_{\bar{v}}\bar{u} = \left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|}\right)\left(\frac{1}{\|\bar{v}\|}\bar{v}\right) = \frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2}\bar{v}$$


$\bar{u}$
$\bar{u} - \bar{v}$
θ
$\bar{v}$


$-\bar{v}$
$\bar{u}$
$\bar{u} + (-\bar{v})$
$= \bar{u} - \bar{v}$

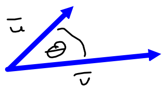
Now, let's talk about scalar components (length of shadow) and projections (The shadow vector)





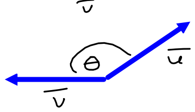
$\pm\ \|\overline{w}\|$ = scalar component (length) of $\overline{u}$ onto $\overline{v}$.

$+\|\overline{w}\| = \text{comp}_{\overline{v}}\overline{u}$



$0 \le \theta \le \frac{\pi}{2}$

$-\|\overline{w}\| = \text{comp}_{\overline{v}}\overline{u}$



$\frac{\pi}{2} \le \theta \le \pi$

* assuming θ is acute.

Let's find $\|\overline{w}\|$:

$$\frac{\|\overline{w}\|}{\|\overline{u}\|} = \cos\theta \implies \|\overline{w}\| = \|\overline{u}\|\cos\theta^{*}$$

$$= \|\overline{u}\|\left(\frac{\overline{u}\cdot\overline{v}}{\|\overline{u}\|\|\overline{v}\|}\right) = \frac{\overline{u}\cdot\overline{v}}{\|\overline{v}\|} = \text{comp}_{\overline{v}}\overline{u}$$

$$\text{Now } \overline{w} = \text{proj}_{\overline{v}}\overline{u} = \underbrace{\underbrace{\left(\frac{\overline{u}\cdot\overline{v}}{\|\overline{v}\|}\right)}_{\text{length}}\underbrace{\left(\frac{1}{\|\overline{v}\|}\overline{v}\right)}_{\text{unit vector in direction of } \overline{v}}}_{\text{Length \& Direction } \checkmark} = \left(\frac{\overline{u}\cdot\overline{v}}{\|\overline{v}\|^2}\right)\overline{v}$$

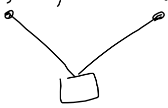
#4 on Test 3 deserves a re-do.

Add a vector question to Test 4

OR

TAKE-HOME

Navigation OR Hanging weight



The book sidesteps the basic techniques in a clever way.

I don't like examples: 7, 5,

EXAMPLE 5 **Finding the Zeros of a Polynomial Function**

Find all the zeros of

$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$

given that $1 + 3i$ is a zero of f.

Skim what they do, but here's the systematic approach.

Recall: to divide $5x^4 - 3x^3 + 2x^2 - 5x + 1$ by $x - 2$

$$\begin{array}{r|rrrrr} 2 & 5 & -3 & 2 & -5 & 1 \\ & & 10 & 14 & 32 & 54 \\ \hline & 5 & 7 & 16 & 27 & 55 \\ & x^3 & x^2 & x & c & r \end{array}$$

This says

$f(x) = (x-2)(5x^3 + 7x^2 + 16x + 27) + 55$

(Shortcut for finding $f(2) = 55$!)

$\boxed{E5}$ Divide $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ by $x - (3+i)$

$$\begin{array}{r|rrrrr} 3+i & 1 & -3 & 6 & 2 & -60 \\ & & 3+i & -1+3i & 12+14i & \\ \hline & 1 & i & 5+3i & 14+14i & \end{array}$$

$i(3+i) = 3i + i^2$

$= -1 + 3i$

$(5+3i)(3+i)$

$= 15 + 5i + 9i + 3i^2$

$= 15 + 14i - 3$

$= 12 + 14i$

$14(1+i)(3+i)$

$= 14(3 + i + 3i + i^2$

It's $1+3i$, jack-ass!

$(a-bi)(a+bi) = a^2 - (bi)^2 = a^2 + b^2 = |a+bi|^2$

$$\begin{array}{r|rrrrr} 1+3i & 1 & -3 & 6 & 2 & -60 \\ & & 1+3i & -11-3i & 4-18i & 60 \\ \hline 1-3i & 1 & -2+3i & -5-3i & 6-18i & 0 \\ & & 1-3i & -1+3i & -6+18i & \\ \hline & 1 & -1 & -6 & 0 & \end{array}$$

0 sweet!

0 sweet!

$(-2+3i)(1+3i)$ Arrg

$= -2 - 6i + 3i + 9i^2$

$= -2 - 3i - 9 = -11 - 3i$

$(-5-3i)(1+3i)$

$= -5 - 15i - 3i - 9i^2$

$= -5 - 18i + 9 = 4 - 18i$

$6(1-3i)(1+3i) = 6(1+9) = 60$!

This says

$f(x) = (x-(1+3i))(x-(1-3i))(x^2 - x - 6)$

$x^2 - x - 6 = (x-3)(x+2)$

$\Rightarrow \boxed{f(x) = (x-(1+3i))(x-(1-3i))(x-3)(x+2)}$

Bonus 5. Let $f(x) = 6x^4 - 25x^3 + 32x^2 + 3x - 10$.

a. (5 pts) Use synthetic division to show that $x = 2 + i$ is a solution of the equation $f(x) = 0$.

b. (5 pts) Find the linear factorization of f that is promised to us in the Fundamental Theorem of Algebra.

a

$2+i$	6	-25	32	3	-10
		$12+6i$	$-32-i$	$1-2i$	10
$2-i$	6	$-13+6i$	$-i$	$4-2i$	0
		$12-6i$	$-2+i$	$-4+2i$	
	6	-1	-2	0	

$(-13+6i)(2+i)$
$= -26-13i+12i+6i^2$
$= -26-i-6 = -32-i$

$-i(2+i) = -2i-i^2 = 1-2i$

$2(2-i)(2+i) = 2(4+1) = 10$

$(x-(2+i))(x-(2-i))(6x^2-x-2)$

$6x^2-4x+3x-2$
$= 2x(3x-2)+1(3x-2) = (3x-2)(2x+1)$

$\Rightarrow f(x) = (x-(2+i))(x-(2-i))(3x-2)(2x+1)$

$6x^2-x-2=0 \Rightarrow$
$a=6, b=-1, c=-2$
$b^2-4ac = (-1)^2-4(6)(-2) = 1+48 = 49 = 7^2$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{1 \pm \sqrt{49}}{2(6)} = \dfrac{1 \pm 7}{12}$ → $\dfrac{8}{12} = \dfrac{2}{3}$, $\dfrac{-6}{12} = -\dfrac{1}{2}$

$\Rightarrow f(x) = 6(x-(2+i))(x-(2-i))(x-\frac{2}{3})(x+\frac{1}{2}) = 6x^4 + \dots$

Don't forget the leading coefficient!

Bonus 6. Let $z = -1 - i$.

a. (5 pts) Find $z + \bar{z}$ and $z\bar{z}$, where $\bar{z}$ is the complex conjugate of z.

b. (5 pts) Express z in trigonometric form.

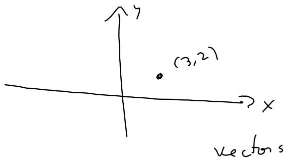
(a) $z = -1 - i \Rightarrow \bar{z} = -1 + i$

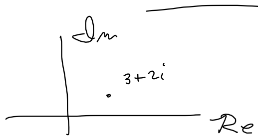
$\Rightarrow z + \bar{z} = -2 = 2\,\mathrm{Re}(z)$

$z\bar{z} = 1^2 + 1^2 = 2 = |z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$

$(-1-i)(-1+i) = 1 - i + i - i^2 = 1 + 1 = 2$

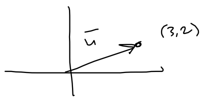
(b) You need to know some 4.3 : Complex Plane



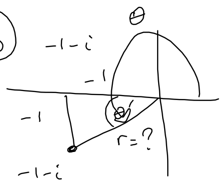


vectors:

$\bar{u} = \langle 3, 2 \rangle$



(b)



$\theta' = 45^\circ$

$\theta = 225^\circ$

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$-1 - i = \sqrt{2}\left(\cos(225^\circ) + i \sin(225^\circ)\right)$

Prefer radians.

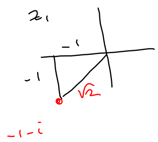
$= \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right)$

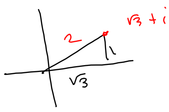
we're getting into §4.3, here.

$z_1 = -1 - i$, $z_2 = \sqrt{3} + i$

$z_1 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$



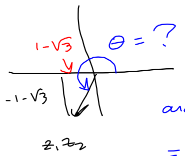


$z_1 z_2 = (-1-i)(\sqrt{3}+i)$

$= -\sqrt{3} - i - i\sqrt{3} - i^2$

$= (-\sqrt{3}+1) + i(-1-\sqrt{3})$

$\frac{5\pi}{4} + \frac{\pi}{6} = \frac{15\pi + 2\pi}{12} = \frac{17\pi}{12}$!



$\arctan\left(\frac{-1-\sqrt{3}}{1-\sqrt{3}}\right)$

$= 75^\circ$!

$180^\circ + 75^\circ = 255^\circ \cdot \frac{\pi}{180}$

$= \frac{17\pi}{12}$

$|z_1 z_2| = \sqrt{(1-\sqrt{3})^2 + (-1-\sqrt{3})^2}$

$= \sqrt{1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3} + 3}$!

$= \sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$!

$(-1-\sqrt{3})^2 = (1+\sqrt{3})^2 = 1 + 2\sqrt{3} + 3$

To multiply complex #s, multiply lengths and add angles!

$\left(\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\right)\left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)$

$= 2\sqrt{2}\left(\cos\left(\frac{5\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{5\pi}{4} + \frac{\pi}{6}\right)\right)$

$r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$

Powers!

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Roots!

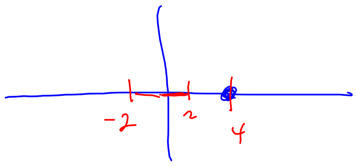
$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)$$

$\sqrt[n]{}$ is PRINCIPLE n^{th} ROOT. There are $n-1$ other n^{th} roots.

$x^2 = 1$

$\sqrt{x^2} = \sqrt{1}$

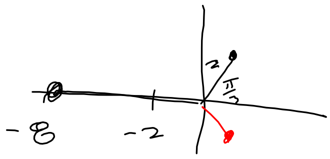
$\Rightarrow x = \pm 1$



$x^3 = 8$

$x = 2$ There are 2 others

$\sqrt[3]{-8} = -2$



$-8 = 8(\cos\pi + i\sin\pi)$

$\sqrt[3]{8}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\frac{2\pi}{3}$ is increment $= \dfrac{2\pi}{\text{index of root}}$

$\frac{\pi}{3} + \frac{2\pi}{3} = \frac{3\pi}{3} = \pi$

$\sqrt[3]{8}\left(\cos\pi + i\sin\pi\right)$

$\pi + \frac{2\pi}{3} = \frac{5\pi}{3}$

$2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$